

Lecture 11. Crash course on conditional expectations

STK2130 – Modelling av stokastiske prosesser



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Conditional expectations and the law of total expectation

Definition

Let A be some event (i.e. we can talk about the probability of A). Then we define a random variable

$$\mathbb{1}_A := \begin{cases} 1, & \text{if } A \text{ happens,} \\ 0, & \text{otherwise.} \end{cases}$$

Definition

Let A be some event such that $\mathbb{P}(A) > 0$ and ξ be a random variable such that $\mathbb{E}|\xi| < \infty$. We will call a conditional expectation of ξ *with respect to the event A* a number

$$\mathbb{E}[\xi \mid A] := \frac{\mathbb{E}[\xi \mathbb{1}_A]}{\mathbb{P}(A)}$$

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Quasi-definition of conditional expectation with respect to a discrete random variable

Let ξ be a random variable such that $\mathbb{E}|\xi| < \infty$ and η be a *discrete* random variable, i.e. there exist a finite or countable sequence $\{x_1, x_2, \dots, x_n, \dots\}$ such that

$$\mathbb{P}(\eta = x_k) > 0, \quad \text{and} \quad \sum_{k=1}^{\infty} \mathbb{P}(\eta = x_k) = 1,$$

i.e. with probability 1 η can take only values from $\{x_1, x_2, \dots, x_n, \dots\}$. A conditional expectation of ξ with respect to η is a random variable (!!!)

$$\mathbb{E}[\xi \mid \eta] = \sum_{k=1}^{\infty} \mathbb{E}[\xi \mid \{\eta = x_k\}] \mathbb{1}_{\{\eta = x_k\}}.$$

Conditional expectations and the law of total expectation

$$\mathbb{E}[\xi | A] := \frac{\mathbb{E}[\xi \mathbb{1}_A]}{\mathbb{P}(A)}, \quad \mathbb{E}[\xi | \eta] = \sum_{k=1}^{\infty} \mathbb{E}[\xi | \{\eta = x_k\}] \mathbb{1}_{\{\eta = x_k\}}.$$

Since $\mathbb{E}[\xi | \eta]$ is a random variable, we can try to compute its expectation.

$$\begin{aligned} \mathbb{E}[\mathbb{E}[\xi | \eta]] &= \mathbb{E}\left[\sum_{k=1}^{\infty} \mathbb{E}[\xi | \{\eta = x_k\}] \mathbb{1}_{\{\eta = x_k\}}\right] = \sum_{k=1}^{\infty} \mathbb{E}[\mathbb{E}[\xi | \{\eta = x_k\}] \mathbb{1}_{\{\eta = x_k\}}] \\ &= \sum_{k=1}^{\infty} \mathbb{E}[\xi | \{\eta = x_k\}] \mathbb{E}[\mathbb{1}_{\{\eta = x_k\}}] = \sum_{k=1}^{\infty} \frac{\mathbb{E}[\xi \mathbb{1}_{\{\eta = x_k\}}]}{\mathbb{P}(\eta = x_k)} \mathbb{P}(\eta = x_k) \\ &= \sum_{k=1}^{\infty} \mathbb{E}[\xi \mathbb{1}_{\{\eta = x_k\}}] = \mathbb{E}\left[\xi \sum_{k=1}^{\infty} \mathbb{1}_{\{\eta = x_k\}}\right] \\ &= \mathbb{E}[\xi] \end{aligned}$$

Conclusion (law of total expectation)

$$\mathbb{E}[\mathbb{E}[\xi | \eta]] = \mathbb{E}[\xi].$$

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$$\mathbb{E}[\xi | A] := \frac{\mathbb{E}[\xi \mathbb{1}_A]}{\mathbb{P}(A)}, \quad \mathbb{E}[\xi | \eta] = \sum_{k=1}^{\infty} \mathbb{E}[\xi | \{\eta = x_k\}] \mathbb{1}_{\{\eta = x_k\}}.$$

Properties of conditional expectation

- If ξ and η are *independent* random variables, then $\mathbb{E}[\xi | \eta] = \mathbb{E}\xi$.
- For any constant $c \in \mathbb{R}$: $\mathbb{E}[c\xi | \eta] = c\mathbb{E}[\xi | \eta]$.
- $\mathbb{E}[\xi_1 + \xi_2 | \eta] = \mathbb{E}[\xi_1 | \eta] + \mathbb{E}[\xi_2 | \eta]$.
- Define a function g by

$$g(x_k) := \mathbb{E}[\xi | \{\eta = x_k\}].$$

Then $\mathbb{E}[\xi | \eta] = g(\eta)$.

Extremal property of conditional expectation

Assume that $\mathbb{E}[\xi^2] < \infty$. Then, for any (measurable) function f such that $\mathbb{E}[f^2(\eta)] < \infty$,

$$\mathbb{E}[(\xi - \mathbb{E}[\xi | \eta])^2] \leq \mathbb{E}[(\xi - f(\eta))^2].$$

Conditional expectations and the law of total expectation

Definition

Let $\mathbb{E}[\xi^2] < \infty$ and η be a discrete random variable. A conditional variance of ξ w.r.t. η is a random variable

$$\text{Var}(\xi \mid \eta) := \mathbb{E} \left[(\xi - \mathbb{E}[\xi \mid \eta])^2 \mid \eta \right]$$

Proposition (law of total variance)

Let $\mathbb{E}[\xi^2] < \infty$ and η be a discrete random variable. Then

$$\text{Var}(\xi) = \mathbb{E}[\text{Var}(\xi \mid \eta)] + \text{Var}(\mathbb{E}[\xi \mid \eta]).$$



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