

Lecture 12. Branching processes. Time reversible Markov chains

STK2130 – Modelling av stokastiske prosesser



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OG ELLIPTISKE FUNKSJONER



Branching Processes

Population with X_0 individuals, each able to produce offspring of the same kind during its lifetime.

$$P_j := \mathbb{P}(\text{An individual produces } j \text{ new offspring}), \quad j = 0, 1, 2, \dots$$

Assumption

All individuals give their offspring independently and

$$P_0 > 0, \quad P_j < 1, \quad j = 0, 1, 2, \dots$$

X_n = Population size in the n th generation, $n = 0, 1, 2, \dots$

$$\mathbb{P}(X_{n+1} = 0 \mid X_n = 0) = 1, \quad \mathbb{P}(X_{n+1} = j \mid X_n = 0) = 0, \quad j \neq 0,$$

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \sum_{s_1, \dots, s_j \geq 0: s_1 + \dots + s_j = i} P_{s_1} P_{s_2} \cdots P_{s_j}, \quad i \neq 0$$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ P_0 & P_1 & P_2 & P_3 & \dots \\ P_0^2 & 2P_0P_1 & 2P_0P_2 + P_1^2 & \dots & \dots \\ P_0^3 & 3P_0^2P_1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Branching Processes

Remark

- Since $P_{0,0} = 1$, then 0 is a recurrent state.
- Since $P_0 > 0$, it follows that $P_{j,0} = P_0^j > 0$. Hence, state j is transient for all $j > 0$.
- Any finite set of transient states $\{1, 2, \dots, n\}$ will be visited only a finite number of times.

Hence, since $P_0 > 0$, the population size converges to 0 or ∞ with probability 1.

Branching Processes

Assume that $X_0 = 1$, and let μ and σ^2 denote respectively the mean and the variance of the number of offspring of an individual. Then:

$$\mu = \sum_{j=0}^{\infty} jP_j, \quad \sigma^2 = \sum_{j=0}^{\infty} (j - \mu)^2 P_j.$$

We also let Z_r be the number of offspring from individual r in the $(n - 1)$ -st generation. Hence:

$$X_n = \sum_{r=1}^{X_{n-1}} Z_r.$$

Branching Processes

$$\begin{aligned}\mathbb{E}[X_n] &= \mathbb{E}[\mathbb{E}[X_n \mid X_{n-1}]] \\ &= \mathbb{E} \left[\mathbb{E} \left[\sum_{r=1}^{X_{n-1}} Z_r \mid X_{n-1} \right] \right] \\ &= \mathbb{E}[X_{n-1}\mu] = \mu\mathbb{E}[X_{n-1}]\end{aligned}$$

Since we have assumed that $X_0 = 1$, it follows by induction that:

$$\mathbb{E}[X_n] = \mu^n.$$

Branching Processes

To find $\text{Var}[X_n]$ we use that:

$$\begin{aligned}\text{Var}[X_n] &= \mathbb{E}[\text{Var}(X_n|X_{n-1})] + \text{Var}[\mathbb{E}(X_n|X_{n-1})] = \mathbb{E}\left[\text{Var}\left(\sum_{r=1}^{X_{n-1}} Z_r \mid X_{n-1}\right)\right] + \text{Var}\left[\mathbb{E}\left(\sum_{r=1}^{X_{n-1}} Z_r \mid X_{n-1}\right)\right] \\&= \mathbb{E}[X_{n-1}\sigma^2] + \text{Var}[X_{n-1}\mu] = \sigma^2\mu^{n-1} + \mu^2 \text{Var}[X_{n-1}] \\&= \sigma^2\mu^{n-1} + \mu^2(\sigma^2\mu^{n-2} + \mu^2 \text{Var}[X_{n-2}]) \\&= \sigma^2\mu^{n-1}(1 + \mu) + \mu^4 \text{Var}[X_{n-2}] \\&= \sigma^2\mu^{n-1}(1 + \mu + \mu^2) + \mu^6 \text{Var}[X_{n-3}] \\&= \dots = \\&= \sigma^2\mu^{n-1}(1 + \mu + \dots + \mu^{n-1}) + \mu^{2n} \text{Var}[X_0].\end{aligned}$$

Hence, since $\text{Var}[X_0] = 0$, we get:

$$\text{Var}[X_n] = \begin{cases} \sigma^2\mu^{n-1} \left(\frac{1-\mu^n}{1-\mu}\right), & \text{if } \mu \neq 1, \\ n\sigma^2, & \text{if } \mu = 1. \end{cases}$$

Branching Processes

We then consider the probability that the population eventually dies out:

$$\pi_0 := \lim_{n \rightarrow \infty} \mathbb{P}(X_n = 0 \mid X_0 = 1).$$

We first note that:

$$\begin{aligned} \mu^n = \mathbb{E}[X_n] &= \mathbb{E}[X_n \mid X_0 = 1] = \sum_{j=1}^{\infty} j \cdot \mathbb{P}(X_n = j \mid X_0 = 1) \\ &\geq \sum_{j=1}^{\infty} 1 \cdot \mathbb{P}(X_n = j \mid X_0 = 1) = 1 - \mathbb{P}(X_n = 0 \mid X_0 = 1). \end{aligned}$$

Hence, it follows that $\mathbb{P}(X_n = 0 \mid X_0 = 1) \geq 1 - \mu^n$. If $\mu < 1$ this implies that:

$$1 \geq \pi_0 = \lim_{n \rightarrow \infty} \mathbb{P}(X_n = 0 \mid X_0 = 1) \geq 1 - \lim_{n \rightarrow \infty} \mu^n = 1 - 0 = 1.$$

Thus, if $\mu < 1$, we must have $\pi_0 = 1$.

Branching Processes

In general, we have:

$$\begin{aligned}\pi_0 &= \mathbb{P}(\text{The population dies out}) \\ &= \sum_{j=0}^{\infty} \mathbb{P}(\text{The population dies out} \mid X_1 = j) P_j \\ &= \sum_{j=0}^{\infty} \pi_0^j P_j.\end{aligned}\tag{1}$$

It can be shown that π_0 is the *smallest positive number* that satisfies (1).

Remark

- Since $\sum_{j=0}^{\infty} P_j = 1$, we see that $\pi_0 = 1$ is always a solution to (1).
- If $P_j = 0$ for all $j \geq 2$, this implies that $\mu < 1$, and hence, $\pi_0 = 1$. Thus, in the following we assume that $P_j > 0$ for at least one $j \geq 2$.

Branching Processes

We then introduce the following functions:

$$\varphi(z) = \sum_{j=0}^{\infty} z^j P_j, \quad \ell(z) = z,$$

and note that a solution z to the equation (1) is found by solving $\varphi(z) = \ell(z)$. We observe that:

$$\varphi(0) = \sum_{j=0}^{\infty} 0^j P_j = P_0 > 0,$$

$$\varphi(1) = \sum_{j=0}^{\infty} 1^j P_j = 1,$$

$$\varphi'(z) = \sum_{j=1}^{\infty} j \cdot z^{j-1} P_j,$$

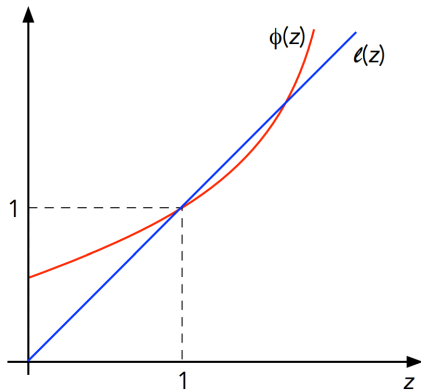
$$\varphi'(1) = \sum_{j=1}^{\infty} j P_j = \mu,$$

$$\varphi''(z) = \sum_{j=2}^{\infty} j(j-1) \cdot z^{j-2} P_j,$$

$$\varphi''(z) > 0 \text{ for all } z > 0.$$

Branching Processes

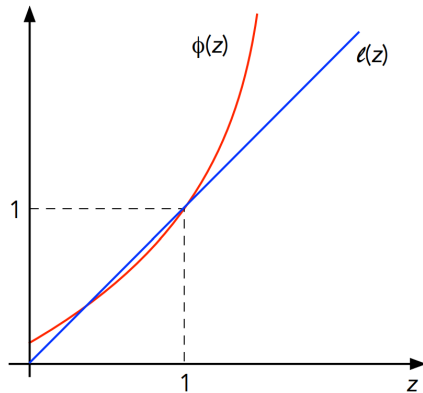
CASE 1. $\varphi'(1) = \mu < 1$



In this case $\phi(z) = \ell(z)$ for $z = 1$ and some $z > 1$. The smallest positive number that satisfies (1) is $\pi_0 = 1$.

Branching Processes

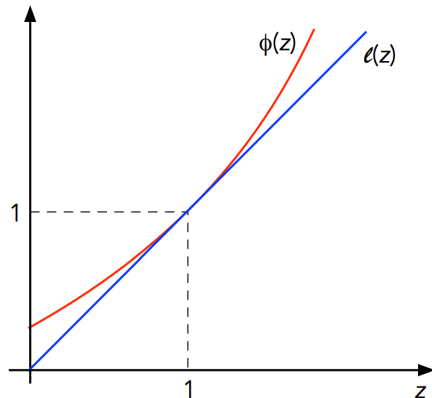
CASE 2. $\varphi'(1) = \mu > 1$



In this case $\phi(z) = \ell(z)$ for $z = 1$ and some $0 < z < 1$. The smallest positive number that satisfies (1) is $\pi_0 < 1$.

Branching Processes

CASE 3. $\varphi'(1) = \mu = 1$



In this case $\phi(z) = \ell(z)$ for $z = 1$ only. The only positive number that satisfies (1) is $\pi_0 = 1$.

Branching Processes

Conclusion

- If $\mu \leq 1$, then $\pi_0 = \mathbb{P}(\text{The population dies out}) = 1$.
- If $\mu > 1$, then $\pi_0 = \mathbb{P}(\text{The population dies out}) < 1$.

Example 4.34

Problem

Assume that

$$P_0 = \frac{1}{2}, \quad P_1 = \frac{1}{4}, \quad P_2 = \frac{1}{4}.$$

Find π_0 .

SOLUTION:

$$\begin{aligned} \mu &= 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 \\ &= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{3}{4} < 1. \end{aligned}$$

Hence, we must have $\pi_0 = 1$.

Example 4.35

Problem

Assume that

$$P_0 = \frac{1}{4}, \quad P_1 = \frac{1}{4}, \quad P_2 = \frac{1}{2}.$$

Find π_0 .

SOLUTION:

$$\begin{aligned} \mu &= 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 \\ &= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} = \frac{5}{4} > 1. \end{aligned}$$

In order to find π_0 we solve (1), which in this case becomes:

$$\pi_0 = \pi_0^0 P_0 + \pi_0^1 P_1 + \pi_0^2 P_2 = \frac{1}{4} + \frac{1}{4} \pi_0 + \frac{1}{2} \pi_0^2$$

or equivalently:

$$2\pi_0^2 - 3\pi_0 + 1 = 2(\pi_0 - 1)(\pi_0 - \frac{1}{2}) = 0$$

Hence, the smallest positive number that satisfies (1) is $\pi_0 = \frac{1}{2}$.

Time Reversible Markov Chains

Time Reversible Markov Chains

Consider a stationary Markov chain with transition probabilities $P_{i,j}$ and stationary probabilities $\pi_i > 0$, $i, j \in \mathcal{S}$ (e.g. ergodic Markov chain with initial distribution equal to $\boldsymbol{\pi}$).

We then consider the backwards chain $X_n, X_{n-1}, X_{n-2}, \dots$

It turns out that the backwards chain is also a time-homogeneous Markov chain with transition probabilities $Q_{i,j}$, $i, j \in \mathcal{S}$ given by:

$$\begin{aligned} Q_{i,j} &= \mathbb{P}(X_m = j \mid X_{m+1} = i) \\ &= \frac{\mathbb{P}(\{X_m = j\} \cap \{X_{m+1} = i\})}{P(X_{m+1} = i)} \\ &= \frac{\mathbb{P}(X_m = j)P(X_{m+1} = i \mid X_m = j)}{\mathbb{P}(X_{m+1} = i)} = \frac{\pi_j P_{j,i}}{\pi_i}. \end{aligned}$$

Time Reversible Markov Chains

Definition

We say that $\{X_n\}$ is time reversible if $Q_{i,j} = P_{i,j}$ for all $i, j \in \mathcal{S}$.

Hence, $\{X_n\}$ is time reversible if and only if:

$$\frac{\pi_j P_{j,i}}{\pi_i} = P_{i,j},$$

for all $i, j \in \mathcal{S}$, or equivalently if and only if:

$$\pi_i P_{i,j} = \pi_j P_{j,i}, \quad \text{for all } i, j \in \mathcal{S}.$$

Time Reversible Markov Chains

Theorem (Time reversible chain)

Assume that, for a stationary Markov chain with a unique stationary distribution, we can find non-negative numbers x_i , $i \in \mathcal{S}$ such that:

$$x_i P_{i,j} = x_j P_{j,i}, \quad \text{for all } i, j \in \mathcal{S} \text{ and } \sum_{i \in \mathcal{S}} x_i = 1. \quad (2)$$

Then the Markov chain is time reversible.

PROOF. If x_i , $i \in \mathcal{S}$ satisfy (2), then it follows that:

$$\sum_{i \in \mathcal{S}} x_i P_{i,j} = x_j \sum_{i \in \mathcal{S}} P_{j,i} = x_j, \quad \text{for all } j \in \mathcal{S} \text{ and } \sum_{i \in \mathcal{S}} x_i = 1. \quad (3)$$

We have proved that the equations (3) have the unique solution:

$$x_i = \pi_i, \quad \text{for all } i \in \mathcal{S},$$

which completes the proof.

Example 4.37

Consider a Markov chain $\{X_n\}$ with state space $\mathcal{S} = \{0, 1, \dots, M\}$ and transition probabilities:

$$P_{i,i+1} = \alpha_i = 1 - P_{i,i-1}, \quad i = 1, \dots, M-1,$$

$$P_{0,1} = \alpha_0 = 1 - P_{0,0},$$

$$P_{M,M} = \alpha_M = 1 - P_{M,M-1}.$$

In matrix form we have

$$\mathbf{P} = \begin{pmatrix} 1 - \alpha_0 & \alpha_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 - \alpha_1 & 0 & \alpha_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 - \alpha_2 & 0 & \alpha_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{M-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 - \alpha_{M-1} & 0 & \alpha_{M-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 - \alpha_M & \alpha_M \end{pmatrix}$$

Example 4.37

In this case the long run rate of transitions from i to $i + 1$ must be equal to the long run rate of transitions from $i + 1$ to i . From this it can be shown that:

$$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}, i = 0, 1, \dots, (M - 1).$$

That is, the Markov chain is time reversible.

In order to find the stationary probabilities we solve the following equations:

$$\begin{aligned}\pi_0 \alpha_0 &= \pi_1 (1 - \alpha_1), \\ \pi_1 \alpha_1 &= \pi_2 (1 - \alpha_2), \\ &\vdots \\ \pi_{M-1} \alpha_{M-1} &= \pi_M (1 - \alpha_M).\end{aligned}$$

Example 4.37

Hence, we get:

$$\pi_1 = \frac{\alpha_0}{1 - \alpha_1} \pi_0,$$

$$\pi_2 = \frac{\alpha_1}{1 - \alpha_2} \pi_1 = \frac{\alpha_1 \alpha_0}{(1 - \alpha_2)(1 - \alpha_1)} \pi_0,$$

$\vdots$

$$\pi_M = \frac{\alpha_{M-1}}{1 - \alpha_M} \pi_{M-1} = \frac{\alpha_{M-1} \cdots \alpha_1 \alpha_0}{(1 - \alpha_M) \cdots (1 - \alpha_2)(1 - \alpha_1)} \pi_0.$$

Example 4.37

We then use that $\sum_{j=0}^M \pi_j = 1$ and get:

$$\pi_0 \left(1 + \sum_{j=1}^M \frac{\alpha_{j-1} \cdots \alpha_1 \alpha_0}{(1 - \alpha_j) \cdots (1 - \alpha_2)(1 - \alpha_1)} \right) = 1.$$

From this it follows that:

$$\pi_0 = \left(1 + \sum_{j=1}^M \frac{\alpha_{j-1} \cdots \alpha_1 \alpha_0}{(1 - \alpha_j) \cdots (1 - \alpha_2)(1 - \alpha_1)} \right)^{-1}$$

and that:

$$\pi_j = \frac{\alpha_{j-1} \cdots \alpha_1 \alpha_0}{(1 - \alpha_j) \cdots (1 - \alpha_2)(1 - \alpha_1)} \pi_0, \quad j = 1, \dots, M.$$

Example 4.37

Assume in particular that $\alpha_j = \alpha$, $j = 0, 1, \dots, M$ and let $\beta = \alpha/(1 - \alpha)$.

We then get:

$$\begin{aligned}\pi_0 &= \left(1 + \sum_{j=1}^M \frac{\alpha^j}{(1 - \alpha)^j} \right)^{-1} \\ &= \left(\frac{1 - \beta^{M+1}}{1 - \beta} \right)^{-1} = \frac{1 - \beta}{1 - \beta^{M+1}}\end{aligned}$$

and

$$\pi_j = \frac{\beta^j(1 - \beta)}{1 - \beta^{M+1}}, \quad j = 1, \dots, M.$$



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