

Lecture 13. Time reversible Markov chains

STK2130 – Modelling av stokastiske prosesser



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Time Reversible Markov Chains

We recall that the time reversibility equations implies that:

$$\begin{aligned}x_i P_{i,j} &= x_j P_{j,i}, \\x_k P_{k,j} &= x_j P_{j,k}.\end{aligned}$$

Assuming that $P_{i,j}P_{j,k} > 0$ these equations imply that:

$$\begin{aligned}x_i &= x_j \frac{P_{j,i}}{P_{i,j}}, \\x_j &= x_k \frac{P_{k,j}}{P_{j,k}}.\end{aligned}$$

Hence,

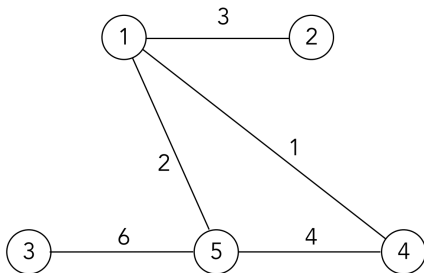
$$\frac{x_i}{x_k} = \frac{P_{k,j}P_{j,i}}{P_{i,j}P_{j,k}}.$$

At the same time the time reversibility equations imply that:

$$\frac{x_i}{x_k} = \frac{P_{k,i}}{P_{i,k}}.$$

Example 4.38

Undirected graph with weighted edges.



Here the transition probabilities are given by:

$$P_{i,j} = \frac{w_{ij}}{\sum_{k \in \mathcal{S}} w_{ik}}, \quad i, j \in \mathcal{S}.$$

Thus, e.g.,

$$P_{1,2} = \frac{3}{6}, \quad P_{1,4} = \frac{1}{6}, \quad P_{1,5} = \frac{2}{6}.$$

Example 4.38

The nodes represent states of a Markov chain with state space $\mathcal{S}$.

Thus, we define:

$X_n =$ The node where the process is at step n , $n = 0, 1, 2, \dots$

We then introduce weights:

$w_{ij} =$ The weight associated with the edge between node i and j , $i, j \in \mathcal{S}$

and let

$$P_{i,j} = \frac{w_{ij}}{\sum_{k \in \mathcal{S}} w_{ik}}, \quad i, j \in \mathcal{S}.$$

Example 4.38

The time reversibility equations:

$$\pi_i P_{i,j} = \pi_j P_{j,i}, \quad i, j \in \mathcal{S}$$

then become

$$\pi_i \frac{w_{ij}}{\sum_{k \in \mathcal{S}} w_{ik}} = \pi_j \frac{w_{ji}}{\sum_{k \in \mathcal{S}} w_{jk}}, \quad i, j \in \mathcal{S}.$$

Since $w_{ij} = w_{ji}$, the equations simplify to:

$$\frac{\pi_i}{\sum_{k \in \mathcal{S}} w_{ik}} = \frac{\pi_j}{\sum_{k \in \mathcal{S}} w_{jk}}, \quad i, j \in \mathcal{S}.$$

which equivalent to:

$$\frac{\pi_i}{\sum_{k \in \mathcal{S}} w_{ik}} = c, \quad i \in \mathcal{S}.$$

Example 4.38

Alternatively, these equations can be written as:

$$\pi_i = c \sum_{k \in \mathcal{S}} w_{ik}, \quad i \in \mathcal{S}.$$

Summing over all i we get:

$$\sum_{i \in \mathcal{S}} \pi_i = c \sum_{i \in \mathcal{S}} \sum_{k \in \mathcal{S}} w_{ik} = 1.$$

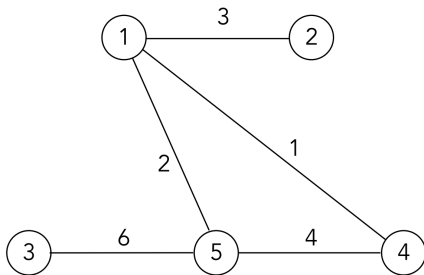
Hence,

$$c = \left(\sum_{i \in \mathcal{S}} \sum_{k \in \mathcal{S}} w_{ik} \right)^{-1}$$

Thus, we get the stationary probabilities:

$$\pi_i = \frac{\sum_{k \in \mathcal{S}} w_{ik}}{\sum_{j \in \mathcal{S}} \sum_{k \in \mathcal{S}} w_{jk}}, \quad i \in \mathcal{S}.$$

Example 4.38



In this graph we get:

$$\pi_1 = \frac{6}{32}, \quad \pi_2 = \frac{3}{32}, \quad \pi_3 = \frac{6}{32}, \quad \pi_4 = \frac{5}{32}, \quad \pi_5 = \frac{12}{32}.$$

Time Reversible Markov Chains

Thus, for a valid solution to the time reversibility equations we must have that:

$$\frac{P_{k,j}P_{j,i}}{P_{i,j}P_{j,k}} = \frac{P_{k,i}}{P_{i,k}}$$

or equivalently:

$$P_{i,k}P_{k,j}P_{j,i} = P_{i,j}P_{j,k}P_{k,i}.$$

Time Reversible Markov Chains

Theorem

A stationary Markov chain for which $P_{i,j} = 0$ whenever $P_{j,i} = 0$ is time reversible if and only if starting in state i , any path back to i has the same probability as the reversed path. That is, if:

$$P_{i,i_1} P_{i_1,i_2} \cdots P_{i_k,i} = P_{i,i_k} P_{i_k,i_{k-1}} \cdots P_{i_1,i}$$

for all states $i, i_1, \dots, i_k$, $k = 1, 2, \dots$

PROOF. That this condition is necessary essentially follows from the argument above. We thus focus on proving sufficiency.

We fix i and j and write the condition in the theorem as:

$$P_{i,i_1} P_{i_1,i_2} \cdots P_{i_k,j} P_{j,i} = P_{i,j} P_{j,i_k} P_{i_k,i_{k-1}} \cdots P_{i_1,i}.$$

By summing over all paths of length $k + 1$ we get that:

$$P_{i,j}^{k+1} P_{j,i} = P_{i,j} P_{j,i}^{k+1}$$

Time Reversible Markov Chains

We then sum over k from 1 to m and divide by m :

$$\frac{P_{j,i} \sum_{k=1}^m P_{i,j}^{k+1}}{m} = \frac{P_{i,j} \sum_{k=1}^m P_{j,i}^{k+1}}{m}.$$

By letting $m \rightarrow \infty$ this implies that:

$$P_{j,i}\pi_j = P_{i,j}\pi_i.$$

Hence, we conclude that the chain is time reversible.

Doubly stochastic transition matrices

Let $\mathbf{P}$ be the transition probability matrix of a Markov chain $\{X_n\}$ with state space $\mathcal{S}$ where $|\mathcal{S}| = n < \infty$. We then know that:

$$\sum_{j \in \mathcal{S}} P_{i,j} \cdot 1 = 1, \quad \text{for all } i \in \mathcal{S}.$$

If $\mathbf{e} = (1, \dots, 1)$ is an n -dimensional vector, we can rewrite this as:

$$\mathbf{P}\mathbf{e}^T = \mathbf{e}.$$

We say that $\mathbf{P}$ is doubly stochastic if we also have:

$$\sum_{i \in \mathcal{S}} P_{i,j} \cdot 1 = 1, \quad \text{for all } j \in \mathcal{S}.$$

In matrix notation this condition can be expressed as:

$$\mathbf{e}\mathbf{P} = \mathbf{e}.$$

Doubly stochastic transition matrices

Proposition

Let $\mathbf{P}$ be the transition probability matrix of an ergodic Markov chain $\{X_n\}$ with state space $\mathcal{S}$ where $|\mathcal{S}| = n < \infty$. Then $\mathbf{P}$ is doubly stochastic if and only if the stationary distribution is uniform, i.e.: $\boldsymbol{\pi} = \frac{1}{n}\mathbf{e}$.

PROOF.

$\mathbf{P}$ is doubly stochastic



$$\mathbf{eP} = \mathbf{e}$$



$$\frac{1}{n}\mathbf{eP} = \frac{1}{n}\mathbf{e}$$



$$\boldsymbol{\pi P} = \boldsymbol{\pi}$$



The stationary distribution is uniform

Doubly stochastic transition matrices

Proposition

Let $\mathbf{P}$ be the transition probability matrix of an ergodic Markov chain $\{X_n\}$ with state space $\mathcal{S}$ where $|\mathcal{S}| = n < \infty$. Assume that $\mathbf{P}$ is doubly stochastic. Then the Markov chain is time reversible if and only if $\mathbf{P}$ is symmetric.

PROOF. Since $\mathbf{P}$ is doubly stochastic, $\pi_i = \frac{1}{n}$ for all $i \in \mathcal{S}$. Hence, we get:

$$Q_{i,j} = \frac{\pi_j P_{j,i}}{\pi_i} = \frac{\frac{1}{n} P_{j,i}}{\frac{1}{n}} = P_{j,i}, \quad i, j \in \mathcal{S}$$

Thus, $\mathbf{Q} = \mathbf{P}$ if and only if $P_{j,i} = P_{i,j}$ for all $i, j \in \mathcal{S}$. That is, $\mathbf{Q} = \mathbf{P}$ if and only if $\mathbf{P}$ is symmetric.

Example

Let $\mathbf{P}$ be the transition probability matrix of a Markov chain $\{X_n\}$ with state space $\mathcal{S} = \{0, 1, 2\}$, where:

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$

Since $\mathbf{P}$ is doubly stochastic and symmetric, the Markov chain is time reversible.

Example

Let $\mathbf{P}$ be the transition probability matrix of a Markov chain $\{X_n\}$ with state space $\mathcal{S} = \{0, 1, 2\}$, where:

$$\mathbf{P} = \begin{pmatrix} 0.0 & 0.1 & 0.9 \\ 0.9 & 0.0 & 0.1 \\ 0.1 & 0.9 & 0.0 \end{pmatrix}$$

Since $\mathbf{P}$ is doubly stochastic and not symmetric, the Markov chain is not time reversible.

Time Reversible Markov Chains

The concept of the reversed chain is useful even when the process is not time reversible.

Proposition

Consider an irreducible Markov chain with transition probability matrix $\mathbf{P}$ and state space S . If we can find positive numbers π , and a transition probability matrix $\mathbf{Q}$ such that:

$$\pi_i P_{i,j} = \pi_j Q_{j,i}, \quad \text{for all } i, j \in S,$$

then $\mathbf{Q}$ is the transition probability matrix of the reversed chain, and π is the stationary distribution for both the original and reversed chain.

Example 4.40

Let L denote the lifetime of a type of light bulb, and assume that the distribution of L in days is:

$$\mathbb{P}(L = i) = p_i, \quad i = 1, 2, \dots,$$

$$\mathbb{E}[L] < \infty.$$

We have an infinite supply of light bulbs of this kind, with lifetimes $L_1, L_2, \dots$ being independent and with the same distribution as L . Each time a bulb fails, it is replaced by a new one the next day.

$$X_n = \text{The age of the light bulb at day } n, \quad n = 1, 2, \dots$$

Then $\{X_n\}$ is a Markov chain with state space $\mathcal{S} = \{1, 2, \dots\}$ and with transition probabilities:

$$P_{i,1} = \frac{\mathbb{P}(L = i)}{\mathbb{P}(L \geq i)} = \frac{p_i}{\sum_{j=i}^{\infty} p_j}, \quad P_{i,i+1} = 1 - P_{i,1}, \quad i = 1, 2, \dots$$

Example 4.40

We then claim that the reversed chain has transition probabilities:

$$\begin{aligned}Q_{i,i-1} &= 1, \quad i > 1 \\ Q_{1,i} &= \mathbb{P}(L = i) = p_i, \quad i \geq 1\end{aligned}$$

To show this we need to find a stationary distribution $\pi_1, \pi_2, \dots$ such that:

$$\pi_i P_{i,j} = \pi_j Q_{j,i}, \quad i, j = 1, 2, \dots$$

We start out by letting $j = 1$ and determine $\pi_1, \pi_2, \dots$ such that:

$$\pi_i P_{i,1} = \pi_i \frac{\mathbb{P}(L = i)}{\mathbb{P}(L \geq i)} = \pi_1 Q_{1,i} = \pi_1 \mathbb{P}(L = i).$$

This is equivalent to:

$$\pi_i = \pi_1 \mathbb{P}(L \geq i).$$

Example 4.40

Summing over all i and using that the π_i -s add up to 1, we get:

$$\begin{aligned}1 &= \sum_{i=1}^{\infty} \pi_i = \pi_1 \sum_{i=1}^{\infty} P(L \geq i) \\&= \pi_1 \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} p_j = \pi_1 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_j \mathbb{1}(j \geq i) \\&= \pi_1 \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p_j \mathbb{1}(i \leq j) = \pi_1 \sum_{j=1}^{\infty} p_j \sum_{i=1}^{\infty} \mathbb{1}(i \leq j) \\&= \pi_1 \sum_{j=1}^{\infty} p_j \cdot j = \pi_1 \mathbb{E}[L].\end{aligned}$$

Hence, it follows that $\pi_1 = (\mathbb{E}[L])^{-1}$, and thus:

$$\pi_i = \pi_1 \mathbb{P}(L \geq i) = \frac{\mathbb{P}(L \geq i)}{\mathbb{E}[L]}, \quad i = 1, 2, \dots$$

Example 4.40

We note that if $j > 1$, we have $P_{i,j} = Q_{j,i} = 0$ for $j \neq i + 1$. Hence, it remains to verify that:

$$\pi_i P_{i,j} = \pi_j Q_{j,i}, \quad i = 1, 2, \dots, j = i + 1.$$

Using the expressions for $\pi_1, \pi_2, \dots$ and that $Q_{i+1,i} = 1$ for $i > 1$, this is equivalent to:

$$\frac{\mathbb{P}(L \geq i)}{\mathbb{E}[L]} \left(1 - \frac{\mathbb{P}(L = i)}{\mathbb{P}(L \geq i)} \right) = \frac{\mathbb{P}(L \geq i + 1)}{\mathbb{E}[L]}.$$

By simplifying this equation we get:

$$\mathbb{P}(L \geq i) - \mathbb{P}(L = i) = \mathbb{P}(L \geq i + 1)$$

which is trivially true.



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