

UiO : **Department of Mathematics**
University of Oslo

Lecture 2. Chapman–Kolmogorov equations

STK2130 – Modelling av stokastiske prosesser



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NIELS HENRIK ABEL
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MATEMATIKER, BERØMT FOR
BANEØYRLENDE ARBEIDER INNEN
LIGNINGSTEORI, UENDELIGE REKKER
OG ELLIPTISKE FUNKSJONER



Brush up: Markov chains

Definition

A discrete time, discrete space stochastic process $\{X_0, X_1, X_2, \dots\}$ is called a *time homogeneous Markov chain* if its *transition probabilities* satisfy the following:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$$

for all states $i_0, i_1, \dots, i_{n-1}, i, j \in \mathcal{S}$ and all $n \geq 0$.

Notation

- $P_{i,j} := \mathbb{P}(X_{n+1} = j \mid X_n = i) = \mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$.
- One-step transition probability matrix: $\mathbf{P} = [P_{i,j}]_{i,j \in \mathcal{S}}$, where $P_{i,j} \geq 0$ for all $i, j \in \mathcal{S}$, and $\sum_{j \in \mathcal{S}} P_{i,j} = 1$ for all $i \in \mathcal{S}$.

n -step transition probabilities

By definition, for a Markov chain, for any $k \geq 0$

$$\mathbb{P}(X_{k+1} = j \mid X_k = i, X_{k-1} = i_{k-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{k+1} = j \mid X_k = i). \quad (1)$$

Is it true that (1) implies also that, for any $k \geq 0$ and $n \geq 1$,

$$\mathbb{P}(X_{k+n} = j \mid X_k = i, X_{k-1} = i_{k-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{k+n} = j \mid X_k = i)?$$

The answer is **YES**.

Exercise

Prove it!

n -step transition probabilities

Notation

- 1 For a homogeneous Markov chain $\{X_n\}$, we denote

$$P_{i,j}^n := \mathbb{P}(X_{k+n} = j \mid X_k = i)$$

– probability that, starting from the state i , the process will get to the state j in n steps.

- 2 It is clear that

$$P_{i,j}^1 = \mathbb{P}(X_{k+1} = j \mid X_k = i) = P_{i,j}.$$

Main problem of the lecture

$\{X_n\}$ – time-homogeneous Markov chain with (discrete) state space $\mathcal{S}$ and transition probability matrix $\mathbf{P} = [P_{i,j}]_{i,j \in \mathcal{S}}$.

Now, fix arbitrary $k \geq 0$ and $n, m \geq 1$.

Problem

Compute

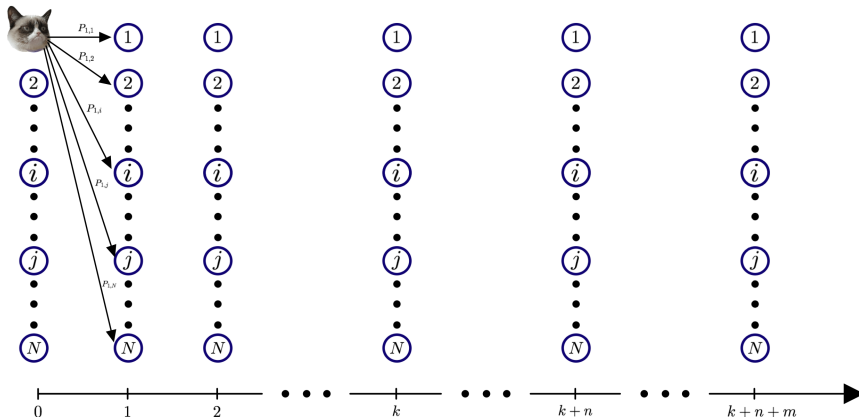
$$P_{i,j}^{n+m} := \mathbb{P}(X_{k+n+m} = j \mid X_k = i).$$

Remark

Note that, due to time-homogeneity of $\{X_n\}$, $P_{i,j}^{n+m}$ does not really depend on k .

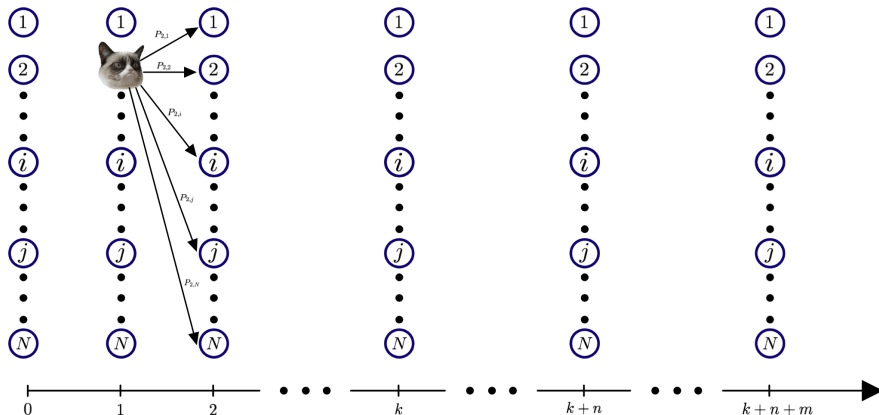
Chapman–Kolmogorov equations: Preface

Let us now see how these probabilities are computed on a particular example. Namely, assume that the state space is finite, i.e. $S = \{1, 2, \dots, N\}$ for some $N \in \mathbb{N}$.

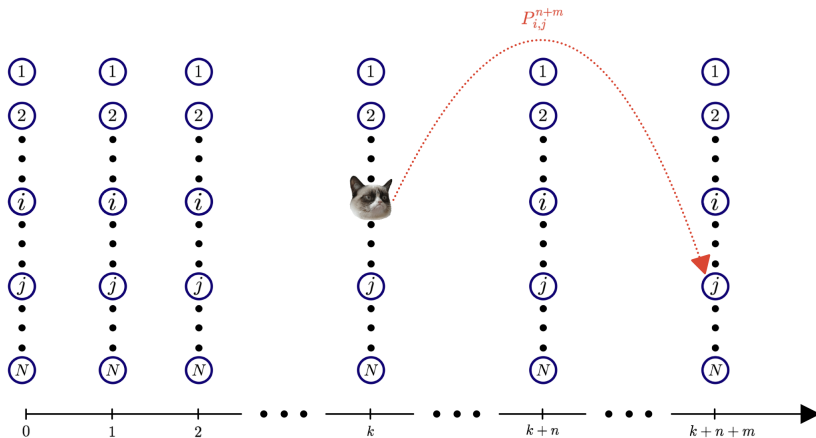


Chapman–Kolmogorov equations: Preface

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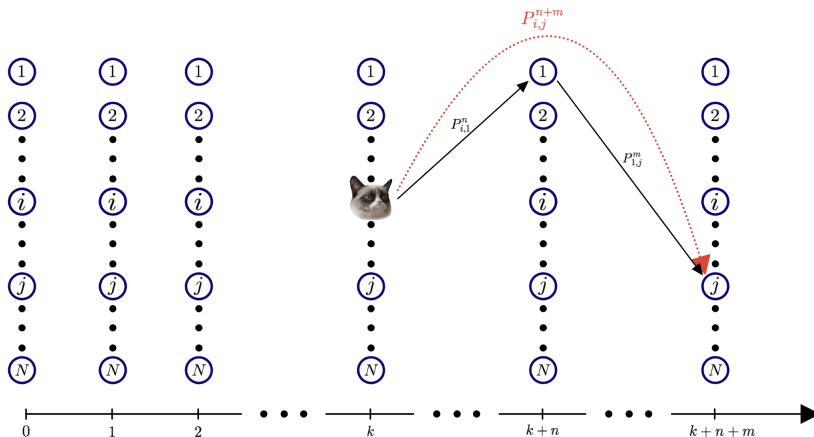


Chapman–Kolmogorov equations: Preface



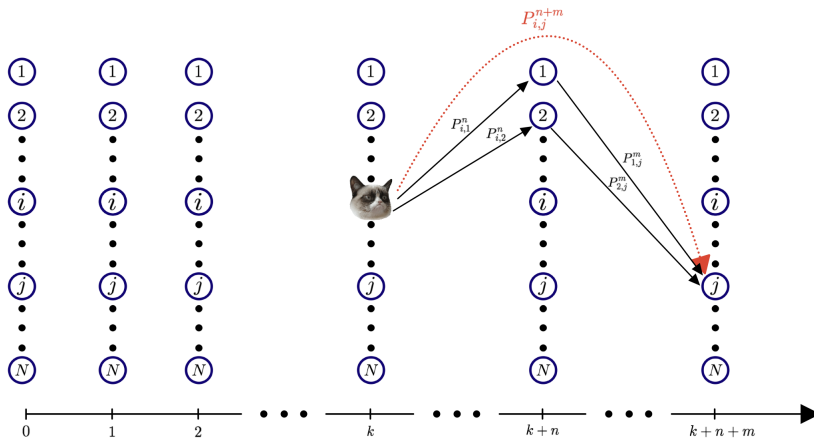
$$\mathbb{P}(X_{k+n+m} = j \mid X_k = i) = P_{i,j}^{n+m} =$$

Chapman–Kolmogorov equations: Preface



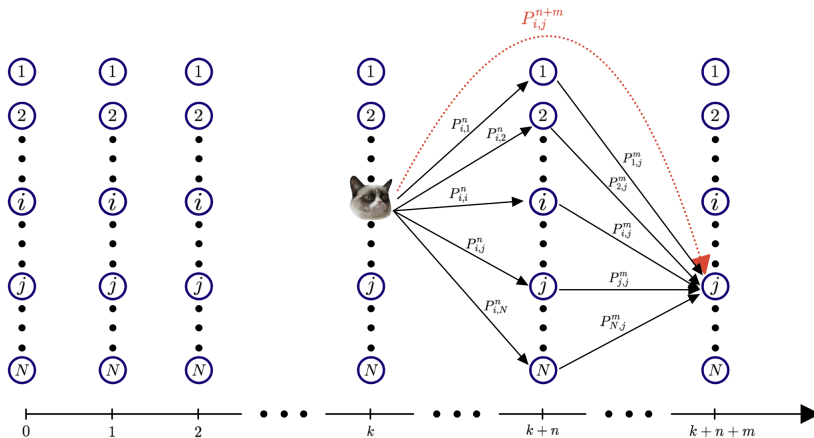
$$\mathbb{P}(X_{k+n+m} = j \mid X_k = i) = P_{i,j}^{n+m} = P_{i,1}^n P_{1,j}^m +$$

Chapman–Kolmogorov equations: Preface



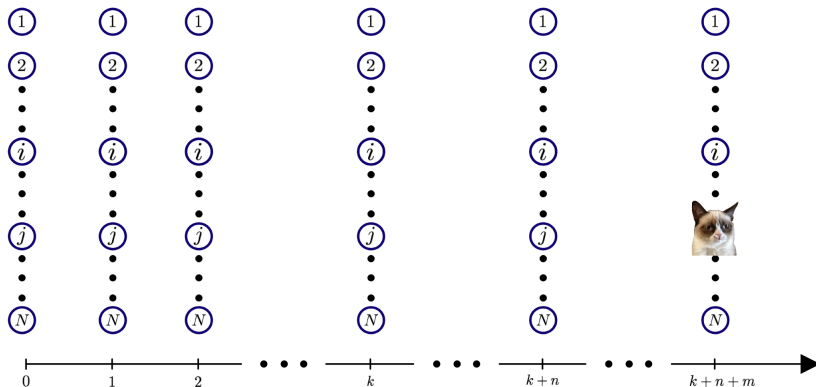
$$\mathbb{P}(X_{k+n+m} = j \mid X_k = i) = P_{i,j}^{n+m} = P_{i,1}^n P_{1,j}^m + P_{i,2}^n P_{2,j}^m +$$

Chapman–Kolmogorov equations: Preface



$$\mathbb{P}(X_{k+n+m} = j \mid X_k = i) = P^{n+m}_{i,j} = P^n_{i,1} P^m_{1,j} + P^n_{i,2} P^m_{2,j} + \dots + P^n_{i,N} P^m_{N,j}$$

Chapman–Kolmogorov equations: Preface



$$\mathbb{P}(X_{k+n+m} = j \mid X_k = i) = P_{i,j}^{n+m} = P_{i,1}^n P_{1,j}^m + P_{i,2}^n P_{2,j}^m + \dots + P_{i,N}^n P_{N,j}^m = \sum_{l \in \mathcal{S}} P_{i,l}^n P_{l,j}^m$$

Chapman–Kolmogorov equations

Chapman–Kolmogorov equations:

$$P_{i,j}^{n+m} = \sum_{l \in \mathcal{S}} P_{i,l}^n P_{l,j}^m, \quad n, m \geq 1, \quad i, j \in \mathcal{S}.$$

More “formal” proof:

$$\begin{aligned} P_{i,j}^{n+m} &= \mathbb{P}(X_{n+m} = j \mid X_0 = i) = \sum_{l \in \mathcal{S}} \mathbb{P}(X_{n+m} = j, X_n = l \mid X_0 = i) \\ &= \sum_{l \in \mathcal{S}} \mathbb{P}(X_{n+m} = j \mid X_n = l, X_0 = i) \mathbb{P}(X_n = l \mid X_0 = i) \\ &= \sum_{l \in \mathcal{S}} \mathbb{P}(X_{n+m} = j \mid X_n = l) \mathbb{P}(X_n = l \mid X_0 = i) \\ &= \sum_{l \in \mathcal{S}} P_{i,l}^n P_{l,j}^m. \end{aligned}$$

Chapman–Kolmogorov equations: matrix form

$$P_{i,j}^{n+m} = \sum_{l \in \mathcal{S}} P_{i,l}^n P_{l,j}^m, \quad n, m \geq 1, \quad i, j \in \mathcal{S}.$$

Denote $\mathbf{P}^{(n)} := [P_{i,j}^n]_{i,j \in \mathcal{S}}$ – the matrix of n -step transition probabilities.

Observation

Chapman–Kolmogorov equations are essentially matrix multiplication formulas:

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \mathbf{P}^{(m)}.$$

In particular,

$$\mathbf{P}^{(2)} = \mathbf{P}^{(1+1)} = \mathbf{P}^{(1)} \mathbf{P}^{(1)} = \mathbf{P} \mathbf{P} = \mathbf{P}^2$$

and, by induction,

$$\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{(n-1)} \mathbf{P}^{(1)} = \mathbf{P}^{n-1} \mathbf{P} = \mathbf{P}^n.$$

Example 4.8

Consider a Markov chain with state space $\mathcal{S} = \{0, 1\}$, where 0 represents rain and 1 represents no rain; X_n – weather on day n . We assume that:

- If it rains today, then it will rain tomorrow with probability 0.75,
- If it does not rain today, then it will rain tomorrow with probability 0.35.

$$\mathbf{P} = \begin{pmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{pmatrix},$$

$$\begin{aligned} \mathbf{P}^{(2)} &= \mathbf{P}^2 = \begin{pmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{pmatrix} \begin{pmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{pmatrix} \\ &= \begin{pmatrix} 0.75 * 0.75 + 0.25 * 0.35 & 0.75 * 0.25 + 0.25 * 0.65 \\ 0.35 * 0.75 + 0.65 * 0.35 & 0.35 * 0.25 + 0.65 * 0.65 \end{pmatrix} \\ &= \begin{pmatrix} 0.65 & 0.35 \\ 0.49 & 0.51 \end{pmatrix}. \end{aligned}$$

Example 4.8

Consider a Markov chain with state space $\mathcal{S} = \{0, 1\}$, where 0 represents rain and 1 represents no rain; X_n – weather on day n . We assume that:

- If it rains today, then it will rain tomorrow with probability 0.75,
- If it does not rain today, then it will rain tomorrow with probability 0.35.

Question

What is the probability that it will rain four days from today given that it is raining today?

$$\mathbf{P} = \begin{pmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{pmatrix}, \quad \mathbf{P}^{(2)} = \begin{pmatrix} 0.65 & 0.35 \\ 0.49 & 0.51 \end{pmatrix}.$$

$$\mathbf{P}^{(4)} = \mathbf{P}^4 = (\mathbf{P}^2)^2 = \begin{pmatrix} 0.5940 & 0.4060 \\ 0.5684 & 0.4316 \end{pmatrix}.$$

Answer: 0.5940.

Example 4.9

Suppose that whether or not it rains today depends on previous weather conditions *through the last two days*:

- If it rained today and yesterday (RR), then the probability of rain tomorrow is 0.7.
- If it rained today but not yesterday ($\bar{R}R$), then the probability of rain tomorrow is 0.5.
- If it did not rain today but it rained yesterday ($R\bar{R}$), then the probability of rain tomorrow is 0.4.
- If it did not rain today and yesterday ($\bar{R}\bar{R}$), then the probability of rain tomorrow is 0.2.

Recall that $\mathcal{S} = \{RR, \bar{R}R, R\bar{R}, \bar{R}\bar{R}\}$, X_n – the weather on days $n - 1$ and n .

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix},$$

where the row and column order is: $RR, \bar{R}R, R\bar{R}, \bar{R}\bar{R}$.

Example 4.9

Problem

What is the probability of rain on Thursday if it rains on Monday and Tuesday?

Row/column order: $RR, \bar{R}R, R\bar{R}, \bar{R}\bar{R}$.

$$\mathbf{P}^{(2)} = \begin{pmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{pmatrix}.$$

Answer: $0.49 + 0.12 = 0.61$.

Example 4.9

Problem

What is the probability that it *does not rain* next Tuesday if it rains on Monday and Tuesday?

Row/column order: $RR, \bar{R}R, R\bar{R}, \bar{R}\bar{R}$.

$$\mathbf{P}^{(7)} = \begin{pmatrix} 0.2723 & 0.1465 & 0.1580 & 0.4233 \\ 0.2633 & 0.1477 & 0.1549 & 0.4340 \\ 0.2441 & 0.1511 & 0.1477 & 0.4571 \\ 0.2352 & 0.1524 & 0.1447 & 0.4678 \end{pmatrix}.$$

Answer: $0.1580 + 0.4233 = 0.5813$.

Example 4.10

An urn always contains 2 balls. Ball colors are **red** and **blue**. At each stage a ball is randomly selected and then replaced by a new ball, which with probability 0.8 is the *same color*, and with probability 0.2 is the *opposite color*, as the ball it replaces.

Problem

If initially both balls are **red**, find the probability that the 5th ball selected is **red**.

Observation

Probability that a selection is **red** is determined by the composition of the urn at the time of the selection.

Example 4.10

An urn always contains 2 balls. Ball colors are red and blue. At each stage a ball is randomly selected and then replaced by a new ball, which with probability 0.8 is the *same color*, and with probability 0.2 is the *opposite color*, as the ball it replaces.

Denote X_n – the number of red balls after the n -th experiment, $X_n \in \{0, 1, 2\}$.

$$P_{0,0} = 0.8, \quad P_{0,1} = 0.2, \quad P_{0,2} = 0,$$

$$P_{1,0} = \mathbb{P}(\text{red ball is selected and replaced}) = 0.5 * 0.2 = 0.1,$$

$$P_{1,1} = \mathbb{P}(\text{any ball is selected and not replaced}) = 0.8,$$

$$P_{1,2} = \mathbb{P}(\text{blue ball is selected and replaced}) = 0.5 * 0.2 = 0.1,$$

$$P_{2,0} = 0, \quad P_{2,1} = 0.2, \quad P_{2,2} = 0.8.$$

Example 4.10

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.1 & 0.8 & 0.1 \\ 0.0 & 0.2 & 0.8 \end{pmatrix}$$

An urn always contains 2 balls. Ball colors are **red** and **blue**. At each stage a ball is randomly selected and then replaced by a new ball, which with probability 0.8 is the *same color*, and with probability 0.2 is the *opposite color*, as the ball it replaces.

Problem

If initially both balls are **red**, find the probability that the 5th ball selected is **red**.

$X_0 = 2 \rightarrow$ 1st selection $\rightarrow X_1 \rightarrow$ 2nd selection $\rightarrow X_2$
 $\rightarrow$ 3rd selection $\rightarrow X_3 \rightarrow$ 4th selection $\rightarrow X_4$
 $\rightarrow$ 5th selection

Example 4.10

$$\mathbf{P}^{(4)} = \begin{pmatrix} 0.4872 & 0.4352 & 0.0776 \\ 0.2176 & 0.5648 & 0.2176 \\ 0.0776 & 0.4352 & 0.4872 \end{pmatrix}$$

$$\begin{aligned} \mathbb{P}(\text{selection 5 is red}) &= \sum_{i=0}^2 \mathbb{P}(\text{selection 5 is red} \mid X_4 = i) \mathbb{P}(X_4 = i) \\ &= \sum_{i=0}^2 \mathbb{P}(\text{selection 5 is red} \mid X_4 = i) \mathbb{P}(X_4 = i \mid X_0 = 2) \\ &= 0 * P_{2,0}^4 + 0.5 * P_{2,1}^4 + 1 * P_{2,2}^4 \\ &= 0.50 * 0.4352 + 0.4872 = 0.7048. \end{aligned}$$

Example 4.10

Problem

If initially we have **red** and **blue** balls, find the probability that the 5th ball selected is **red**.

$$\mathbf{P}^{(4)} = \begin{pmatrix} 0.4872 & 0.4352 & 0.0776 \\ 0.2176 & 0.5648 & 0.2176 \\ 0.0776 & 0.4352 & 0.4872 \end{pmatrix}$$

$$\begin{aligned} \mathbb{P}(\text{selection 5 is red}) &= \sum_{i=0}^2 \mathbb{P}(\text{selection 5 is red} \mid X_4 = i) \mathbb{P}(X_4 = i) \\ &= \sum_{i=0}^2 \mathbb{P}(\text{selection 5 is red} \mid X_4 = i) \mathbb{P}(X_4 = i \mid X_0 = 1) \\ &= 0 * P_{1,0}^4 + 0.5 * P_{1,1}^4 + 1 * P_{1,2}^4 = 0.50 * 0.5648 + 0.2176 = 0.5. \end{aligned}$$

Example 4.10

Problem

If initially we have two **red** balls, find the probability that the 31st ball selected is **red**.

$$\mathbf{P}^{(30)} = \begin{pmatrix} 0.25 & 0.50 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.50 & 0.25 \end{pmatrix}$$

$$\begin{aligned} \mathbb{P}(\text{selection 31 is red}) &= \sum_{i=0}^2 \mathbb{P}(\text{selection 5 is red} \mid X_{30} = i) \mathbb{P}(X_{30} = i) \\ &= \sum_{i=0}^2 \mathbb{P}(\text{selection 5 is red} \mid X_{30} = i) \mathbb{P}(X_{30} = i \mid X_0 = 1) \\ &= 0 * P_{2,0}^{30} + 0.5 * P_{2,1}^{30} + 1 * P_{2,2}^{30} = 0.5 * 0.5 + 0.25 = 0.5. \end{aligned}$$

Example 4.11

Problem

Suppose that balls are successively distributed among **8 urns**, with each ball being equally likely to be put in any of these urns. What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed?

Straightforward way: we introduce Markov chain $\{X_n\}$, where X_n – number of nonempty urns after n distribution, $X_n \in \{0, 1, \dots, 8\}$. In particular $X_0 = 0$.

It is easy to see that transition probabilities are:

$$P_{i,i} = \frac{i}{8}, \quad P_{i,i+1} = \frac{8-i}{8}, \quad i = 0, 1, \dots, 8.$$

Example 4.11: straightforward way

Problem

Suppose that balls are successively distributed among **8 urns**, with each ball being equally likely to be put in any of these urns. What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed?

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/8 & 6/8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/8 & 5/8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/8 & 4/8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6/8 & 2/8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7/8 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now one could find $\mathbf{P}^9$ and write $P_{0,3}^9 = 0.00756$.

Example 4.11: smart way

Problem

Suppose that balls are successively distributed among **8 urns**, with each ball being equally likely to be put in any of these urns. What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed?

X_n – number of nonempty urns after n distributions.

The first transition is deterministic ($0 \rightarrow 1$). Moreover, state 0 cannot be achieved after the first transition. Why do we need it then? If we define

$$Y_n := X_{n+1}, \quad n = 0, 1, 2, \dots,$$

then $\{Y\}$ will have a “smaller” state space $\{1, 2, \dots, 8\}$ and

$$\mathbb{P}(X_9 = 3 \mid X_0 = 0) = \mathbb{P}(Y_8 = 3 \mid Y_0 = 1).$$

Example 4.11: even smarter way

Problem

Suppose that balls are successively distributed among **8 urns**, with each ball being equally likely to be put in any of these urns. What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed?

X_n – number of nonempty urns after n distributions, $Y_n := X_{n+1}$.

We are not really interested in states 5, 6, 7, 8: for our purposes, it is sufficient to know whether there are 1, 2, 3 or ≥ 4 nonempty urns!

Define

$$Z_n := \min\{Y_n, 4\}$$

– states $\{4, 5, 6, 7, 8\}$ of Y_n collapse into one single state 4 of Z_n !

Example 4.11: even smarter way

Problem

Suppose that balls are successively distributed among **8 urns**, with each ball being equally likely to be put in any of these urns. What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed?

X_n – number of nonempty urns after n distributions, $Y_n := X_{n+1}$, $Z_n := \min\{Y_n, 4\}$.

The transition probability matrix of $\{Z\}$ can be easily calculated: that is

$$\mathbf{Q} := \begin{pmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 4.11: even smarter way

Problem

Suppose that balls are successively distributed among **8 urns**, with each ball being equally likely to be put in any of these urns. What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed?

X_n – number of nonempty urns after n distributions, $Y_n := X_{n+1}$, $Z_n := \min\{Y_n, 4\}$.

$$\mathbf{Q}^{(4)} := \begin{pmatrix} 0.0002 & 0.0256 & 0.2563 & 0.7178 \\ 0.0000 & 0.0039 & 0.0952 & 0.9009 \\ 0.0000 & 0.0000 & 0.0198 & 0.9802 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

In order to get the result, we can compute $\mathbf{Q}^{(8)} = \mathbf{Q}^{(4)}\mathbf{Q}^{(4)}$ or use Chapman–Kolmogorov equations and find

$$\mathbb{P}(Z_8 = 3 \mid Z_0 = 1) := Q_{1,3}^8 = \sum_{k=1}^4 Q_{1,k}^4 Q_{k,3}^4 = 0.00756.$$

Unconditional probabilities

So far, all of the probabilities we have considered are *conditional probabilities*. For instance, $P_{i,j}^n$ is the probability that the state at time n is j *given that the initial state at time 0 is i* .

Question

What if the starting state X_0 is *not* deterministic and we want to compute unconditional probabilities (e.g. $\mathbb{P}(X_n = j)$)?

Unconditional probabilities

If the unconditional distribution of the state at time n is desired, *it is necessary to specify the probability distribution of the initial state.*

Denote

$$\pi_i^{(n)} := \mathbb{P}(X_n = i), \quad i \in \mathcal{S}, \quad n \geq 0.$$

Then, for any $n, k \geq 0$,

$$\pi_j^{(n+k)} = \mathbb{P}(X_{n+k} = j) = \sum_{i \in \mathcal{S}} \mathbb{P}(X_{n+k} = j \mid X_0 = i) \mathbb{P}(X_n = i) = \sum_{i \in \mathcal{S}} P_{i,j}^k \pi_i^{(n)}.$$

If we define a (line) vector

$$\boldsymbol{\pi}^{(n)} := (\pi_i^{(n)})_{i \in \mathcal{S}},$$

then

$$\boldsymbol{\pi}^{(n+k)} = \boldsymbol{\pi}^{(n)} \mathbf{P}^{(k)}.$$

In particular,

$$\boldsymbol{\pi}^{(k)} = \boldsymbol{\pi}^{(0)} \mathbf{P}^{(k)}.$$

Example 4.8 revisited

Consider a Markov chain with state space $\mathcal{S} = \{0, 1\}$, where 0 represents rain and 1 represents no rain; X_n – weather on day n . We assume that:

- If it rains today, then it will rain tomorrow with probability 0.75,
- If it does not rain today, then it will rain tomorrow with probability 0.35.

Question

What is the probability that it will rain four days from today, if it rains today with probability 0.4?

$$\mathbf{P} = \begin{pmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{pmatrix}, \quad \mathbf{P}^{(4)} = \begin{pmatrix} 0.5940 & 0.4060 \\ 0.5684 & 0.4316 \end{pmatrix}.$$

Then

$$\mathbb{P}(X_4 = 0) = \pi_0 P_{0,0}^4 + \pi_1 P_{1,0}^4 = 0.4 * 0.5940 + 0.6 * 0.5684 = 0.57864.$$

Unconditional probabilities

Now, assume that it happens that there exists a limit

$$\lim_{n \rightarrow \infty} \pi_i^{(n)} = \pi_i, \quad i \in \mathcal{S},$$

and denote

$$\boldsymbol{\pi} = (\pi_i)_{i \in \mathcal{S}}.$$

Assume that $\mathcal{S} = \{1, \dots, N\}$. Then we can write:

$$\pi_j = \lim_{n \rightarrow \infty} \pi_j^{(n+1)} = \lim_{n \rightarrow \infty} \sum_{i=1}^N \pi_i^{(n)} P_{i,j} = \sum_{i=1}^N \lim_{n \rightarrow \infty} \pi_i^{(n)} P_{i,j} = \sum_{i=1}^N \pi_i P_{i,j},$$

i.e. $\boldsymbol{\pi}$ must satisfy

$$\boldsymbol{\pi} = \boldsymbol{\pi} \mathbf{P}.$$

Also observe that each $\pi_i \in [0, 1]$ and $\sum_{i=1}^N \pi_i = 1$ (why?) – $\boldsymbol{\pi}$ is a probability distribution!

Example 4.8 revisited for the second time

Consider a Markov chain with state space $\mathcal{S} = \{0, 1\}$, where 0 represents rain and 1 represents no rain; X_n – weather on day n . We assume that:

- If it rains today, then it will rain tomorrow with probability 0.75,
- If it does not rain today, then it will rain tomorrow with probability 0.35.

$$\mathbf{P} = \begin{pmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{pmatrix}.$$

Let us find a probability distribution $\boldsymbol{\pi} = (\pi_0, \pi_1)$ such that $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ (*it exists!*)

$$\begin{cases} \pi_0 = 0.75\pi_0 + 0.35\pi_1, \\ \pi_1 = 0.25\pi_0 + 0.65\pi_1. \end{cases}$$

By inserting $\pi_1 = 1 - \pi_0$ into the first equation, we get:

$$\pi_0 = 0.75\pi_0 + 0.35(1 - \pi_0) = 0.4\pi_0 + 0.35,$$

and thus

$$\pi_0 = \frac{0.35}{0.6} \approx 0.5833, \quad \pi_1 = 1 - \pi_0 \approx 0.4167.$$

Example 4.8 revisited for the second time

Consider a Markov chain with state space $\mathcal{S} = \{0, 1\}$, where 0 represents rain and 1 represents no rain; X_n – weather on day n . We assume that:

- If it rains today, then it will rain tomorrow with probability 0.75,
- If it does not rain today, then it will rain tomorrow with probability 0.35.

$$\pi_0 \approx 0.5833, \quad \pi_1 \approx 0.4167.$$

$$\mathbf{P} = \begin{pmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{pmatrix}, \quad \mathbf{P}^{(2)} = \begin{pmatrix} 0.65 & 0.35 \\ 0.49 & 0.51 \end{pmatrix}, \quad \mathbf{P}^{(4)} = \begin{pmatrix} 0.5940 & 0.4060 \\ 0.5684 & 0.4316 \end{pmatrix}$$

$$\mathbf{P}^{(12)} = \begin{pmatrix} 0.5833 & 0.4167 \\ 0.5833 & 0.4167 \end{pmatrix}.$$

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**Lecture 2. Chapman–Kolmogorov
equations**

STK2130 – Modelling av stokastiske prosesser

