

UiO : **Department of Mathematics**
University of Oslo

Lecture 4. First passage probabilities. Classification of states

STK2130 – Modelling av stokastiske
prosesser



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MATEMATIKER, BERØMT FOR
BANEØRYTENDE ARBEIDER INNEN
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OG ELLIPTISKE FUNKSJONER



Brush-up: First passage probabilities

Consider a Markov chain $\{X_n\}$ with state space S and transition probability matrix $\mathbf{P}$, and let $\mathcal{A}$ be a non-empty proper subset of S .

We want to calculate the following probability:

$$\beta := \mathbb{P}(X_k \in \mathcal{A} \text{ for some } 1 \leq k \leq m \mid X_0 = i),$$

where we assume that $i \notin \mathcal{A}$. In other words, we are interested in probability that $\{X\}$ will visit $\mathcal{A}$ at least once within m steps given that it starts in i .

In order to analyze this, we introduce:

$$N := \min\{n > 0 : X_n \in \mathcal{A}\}.$$

Thus, N represents the first time the Markov chain enters $\mathcal{A}$.

First passage probabilities: Transition probabilities of $\{W_n\}$

$$\beta := \mathbb{P}(X_k \in \mathcal{A} \text{ for some } 1 \leq k \leq m \mid X_0 = i), \quad W_n := \begin{cases} X_n, & \text{if } n < N, \\ A, & \text{if } n \geq N. \end{cases}$$

The transition probability matrix of $\{W_n\}$, denoted $\mathbf{Q} = (Q_{i,j})$, is given by:

- if $i \notin \mathcal{A}, j \notin \mathcal{A}$,

$$Q_{i,j} = P_{i,j}.$$

- if $i \notin \mathcal{A}$,

$$Q_{i,A} = \sum_{j \in \mathcal{A}} P_{i,j}.$$

- $Q_{A,A} = 1$.

We now have:

$$\beta = \mathbb{P}(X_k \in \mathcal{A} \text{ for some } 1 \leq k \leq m \mid X_0 = i) = \mathbb{P}(W_m = A \mid W_0 = i) = Q_{i,A}^m.$$

Example 4.13

Let $\{X_n\}$ be a Markov chain with the state set $S = \{0, 1, 2, 3\}$ and transition probability matrix $\mathbf{P}$. We then introduce:

$$N = \min\{n \geq 4 : X_{n-3} = 1, X_{n-2} = 2, X_{n-1} = 1, X_n = 2\}$$

Thus, N is the number of steps we need to observe the sequence $(1, 2, 1, 2)$ for the first time.

Problem

For a given value of $k \geq 4$ calculate $\mathbb{P}(N \leq k)$ given that $X_0 = 0$.

Example 4.13

To solve this problem, we start out by introducing the following process $\{Z_n\}$:

Z_n = The minimal possible number of steps required to observe the first $(1, 2, 1, 2)$

for any given time step n of the original process. The state space of $\{Z_n\}$ is $\mathcal{T} = \{0, 1, 2, 3, 4\}$ and we observe that:

$$\begin{aligned}\mathbb{P}(Z_{n+1} = 0 \mid Z_n = 0) &= 1, & \mathbb{P}(Z_{n+1} = 0 \mid Z_n = 1) &= P_{1,2}, \\ \mathbb{P}(Z_{n+1} = 1 \mid Z_n = 2) &= P_{2,1}, & \mathbb{P}(Z_{n+1} = 2 \mid Z_n = 3) &= P_{1,2}.\end{aligned}$$

Determining $\mathbb{P}(Z_{n+1} = 3 \mid Z_n = 4)$ is more difficult, but we may express this by conditioning on X_n as follows:

$$\mathbb{P}(Z_{n+1} = 3 \mid Z_n = 4) = \sum_{i \in \{0,2,3\}} \mathbb{P}(Z_{n+1} = 3 \mid Z_n = 4, X_n = i) \mathbb{P}(X_n = i)$$

The term $\mathbb{P}(Z_{n+1} = 3 \mid Z_n = 4, X_n = 1) \mathbb{P}(X_n = 1)$ is skipped since $X_n = 1$ implies that $Z_n < 4$.

Example 4.13

The problem with the transition probability $\mathbb{P}(Z_{n+1} = 3|Z_n = 4)$ is that it depends on n . In order to avoid this problem, we need a process which contains more information about the state of the original process.

Thus, we introduce a new Markov chain $\{Y_n\}$ defined as follows:

$$Y_n = Z_n + X_n \mathbb{1}_{\{Z_n=4\}}$$

The state space of $\{Y_n\}$ is $\mathcal{T}_1 = \{0, 1, 2, 3, 4, 6, 7\}$.

Remark

$\mathcal{T}_1$ does not contain the state 5 since the combination where $Z_n = 4$ and $X_n = 1$ is not possible.

Remark

If $Y_n \geq 4$, then $X_n = Y_n - 4$. Thus, in this case the state of the original process can be derived from Y_n . As a result the transition probabilities of $\{Y_n\}$ do not depend on n .

Example 4.13

The transition probability matrix of $\{Y_n\}$ with state space $\mathcal{T} = \{0, 1, 2, 3, 4, 6, 7\}$ is:

$$\mathbf{Q} = \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{1,2} & 0 & 0 & P_{1,1} & P_{1,0} & 0 & P_{1,3} \\ 0 & P_{2,1} & 0 & 0 & P_{2,0} & P_{2,2} & P_{2,3} \\ 0 & 0 & P_{1,2} & P_{1,1} & P_{1,0} & 0 & P_{1,3} \\ \hline 0 & 0 & 0 & P_{0,1} & P_{0,0} & P_{0,2} & P_{0,3} \\ 0 & 0 & 0 & P_{2,1} & P_{2,0} & P_{2,2} & P_{2,3} \\ 0 & 0 & 0 & P_{3,1} & P_{3,0} & P_{3,2} & P_{3,3} \end{array} \right)$$

If $Y_n = 1$ or $Y_n = 3$, then $X_n = 1$. If $Y_n = 2$, then $X_n = 2$.

The entries below the horizontal dashed line corresponds to cases where $Y_n \geq 4$, implying that $X_n = Y_n - 4$.

Example 4.13

Since we have assumed that $X_0 = 0$, it follows that $Y_0 = 4$. Moreover, recalling that:

$$N = \min\{n \geq 4 : X_{n-3} = 1, X_{n-2} = 2, X_{n-1} = 1, X_n = 2\}$$

and using the fact that state 0 is an absorbing state for $\{Y_n\}$, it follows that

$$N \leq k \iff Y_k = 0$$

Hence, we get that:

$$\mathbb{P}(N \leq k) = \mathbb{P}(Y_k = 0 \mid Y_0 = 4) = Q_{4,0}^k$$

Remark

In the textbook, the numbering of states is different.

First-passage probabilities

Consider again a Markov chain $\{X_n\}$ with state space $\mathcal{S}$ and transition probability matrix $\mathbf{P}$, and let $\mathcal{A}$ be a non-empty proper subset of $\mathcal{S}$.

We now consider the following probability:

$$\alpha := \mathbb{P}(X_k \notin \mathcal{A} \text{ for all } 1 \leq k \leq m-1, X_m = j \mid X_0 = i)$$

As before, we let $N := \min\{n > 0 : X_n \in \mathcal{A}\}$ and:

$$W_n := \begin{cases} X_n, & \text{if } n < N, \\ A, & \text{if } n \geq N. \end{cases}$$

As before, we let the state A be some arbitrary state in $\mathcal{A}$, and hence, the state space of $\{W_n\}$ becomes $(\mathcal{S} \setminus \mathcal{A}) \cup \{A\}$.

First-passage probabilities. Case 1: $i \notin \mathcal{A}, j \notin \mathcal{A}$

Let $i \notin \mathcal{A}, j \notin \mathcal{A}$.

In this case the event $\{X_k \notin \mathcal{A} \text{ for all } 1 \leq k \leq m-1, X_m = j\}$ is equivalent to the event $\{W_m = j\}$.

Hence, it follows that:

$$\alpha = \mathbb{P}(W_m = j \mid W_0 = i) = Q_{i,j}^m.$$

First-passage probabilities. Case 2: $i \notin \mathcal{A}, j \in \mathcal{A}$

Let $i \notin \mathcal{A}, j \in \mathcal{A}$

In this case we have:

$$\begin{aligned}\alpha &= \sum_{r \notin \mathcal{A}} \mathbb{P}(X_k \notin \mathcal{A} \text{ for all } 1 \leq k \leq m-2, X_{m-1} = r, X_m = j \mid X_0 = i) \\ &= \sum_{r \notin \mathcal{A}} \mathbb{P}(X_k \notin \mathcal{A} \text{ for all } 1 \leq k \leq m-2, X_{m-1} = r \mid X_0 = i) \times \\ &\quad \times \mathbb{P}(X_m = j \mid X_0 = i, X_k \notin \mathcal{A} \text{ for all } 1 \leq k \leq m-2, X_{m-1} = r) \\ &= \sum_{r \notin \mathcal{A}} \mathbb{P}(X_k \notin \mathcal{A} \text{ for all } 1 \leq k \leq m-2, X_{m-1} = r \mid X_0 = i) \mathbb{P}(X_m = j \mid X_{m-1} = r) \\ &= \sum_{r \notin \mathcal{A}} Q_{i,r}^{m-1} P_{r,j}.\end{aligned}$$

$$\alpha = \sum_{r \notin \mathcal{A}} Q_{i,r}^{m-1} P_{r,j}.$$

First-passage probabilities. Case 3: $i \in \mathcal{A}, j \notin \mathcal{A}$

Let $i \in \mathcal{A}, j \notin \mathcal{A}$

In this case we have:

$$\begin{aligned}\alpha &= \sum_{r \notin \mathcal{A}} \mathbb{P}(X_1 = r, X_k \notin \mathcal{A} \text{ for all } 2 \leq k \leq m-1, X_m = j \mid X_0 = i) \\&= \sum_{r \notin \mathcal{A}} \mathbb{P}(X_1 = r \mid X_0 = i) \times \\&\quad \times \mathbb{P}(X_k \notin \mathcal{A} \text{ for all } 2 \leq k \leq m-1, X_m = j \mid X_0 = i, X_1 = r) \\&= \sum_{r \notin \mathcal{A}} \mathbb{P}(X_1 = r \mid X_0 = i) \mathbb{P}(X_k \notin \mathcal{A} \text{ for all } 2 \leq k \leq m-1, X_m = j \mid X_1 = r) \\&= \sum_{r \notin \mathcal{A}} P_{i,r} Q_{r,j}^{m-1}.\end{aligned}$$

$$\alpha = \sum_{r \notin \mathcal{A}} P_{i,r} Q_{r,j}^{m-1}.$$

First-passage probabilities. Case 4: $i \in \mathcal{A}, j \in \mathcal{A}$

Let $i \in \mathcal{A}, j \in \mathcal{A}$

By combining the previous arguments, we get:

$$\alpha = \sum_{r, s \notin \mathcal{A}} P_{i,r} Q_{r,s}^{m-2} P_{s,j}.$$

Example 4.14

Let $\{X_n\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4, 5\}$, and let $\mathcal{A} = \{3, 4, 5\}$.

We now consider the following probability:

$$\alpha = \mathbb{P}(X_k \notin \mathcal{A} \text{ for all } 1 \leq k \leq 3, X_4 = 2 \mid X_0 = 1).$$

We then let $N = \min\{n > 0 : X_n \in \mathcal{A}\}$ and:

$$W_n = \begin{cases} X_n, & \text{if } n < N, \\ 3, & \text{if } n \geq N. \end{cases}$$

The state space of $\{W_n\}$ becomes $\mathcal{T} = \{1, 2, 3\}$.

Example 4.14

The transition probability matrix of $\{W_n\}$, denoted $\mathbf{Q}$, is given by:

$$Q_{i,j} = P_{i,j}, \quad i, j = 1, 2,$$

$$Q_{i,3} = 1 - P_{i,1} - P_{i,2}, \quad i = 1, 2,$$

$$Q_{3,3} = 1.$$

We then assume that:

$$P_{1,1} = 0.3, \quad P_{1,2} = 0.3, \quad P_{2,1} = 0.1, \quad P_{2,2} = 0.2.$$

Example 4.14

Hence we get that

$$\mathbf{Q} = \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.1 & 0.2 & 0.7 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}$$

From this we calculate that:

$$\mathbf{Q}^{(4)} = \begin{pmatrix} 0.0219 & 0.0285 & 0.9496 \\ 0.0095 & 0.0124 & 0.9781 \\ 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

Hence, we get that:

$$\alpha = \mathbb{P}(X_k \notin \mathcal{A} \text{ for all } 1 \leq k \leq 3, X_4 = 2 \mid X_0 = 1) = P(W_4 = 2 \mid W_0 = 1) = Q_{1,2}^4 = 0.0285$$

Section 4.3. Classification of states

Agreement

$$P_{i,i}^0 = 1$$

Accessible states

Let $\{X_n\}$ be a Markov chain with state space $\mathcal{S}$ and transition probability matrix $\mathbf{P}$.

Definition

State j is said to be accessible from state i , denoted as $i \rightarrow j$, if $P_{i,j}^n > 0$ for some $n \geq 0$.

Note that we have:

$$P_{i,j}^n \leq \mathbb{P}(\cup_{k=0}^{\infty} \{X_k = j\} \mid X_0 = i), \quad n = 0, 1, 2, \dots$$

Hence, we also have:

$$\begin{aligned} \sup_{n \geq 0} P_{i,j}^n &\leq \mathbb{P}\left(\bigcup_{k=0}^{\infty} \{X_k = j\} \mid X_0 = i\right) \\ &\leq \sum_{k=0}^{\infty} \mathbb{P}(X_k = j \mid X_0 = i) = \sum_{k=0}^{\infty} P_{i,j}^k. \end{aligned}$$

Accessible states

If $i \rightarrow j$, then by definition $P_{i,j}^n > 0$ for some $n \geq 0$, and hence we obviously also have that $\sup_n P_{i,j}^n > 0$. This implies that:

$$\mathbb{P} \left(\bigcup_{k=0}^{\infty} \{X_k = j\} \mid X_0 = i \right) \geq \sup_{n \geq 0} P_{i,j}^n > 0.$$

Conversely, if $\mathbb{P} \left(\bigcup_{k=0}^{\infty} \{X_k = j\} \mid X_0 = i \right) > 0$, then also

$$\sum_{k=0}^{\infty} P_{i,j}^k \geq \mathbb{P} \left(\bigcup_{k=0}^{\infty} \{X_k = j\} \mid X_0 = i \right) > 0$$

which implies that $P_{i,j}^n > 0$ for some $n \geq 0$, i.e. $i \rightarrow j$. Therefore we conclude:

$$i \rightarrow j \iff \mathbb{P} \left(\bigcup_{k=0}^{\infty} \{X_k = j\} \mid X_0 = i \right) > 0.$$

State diagram

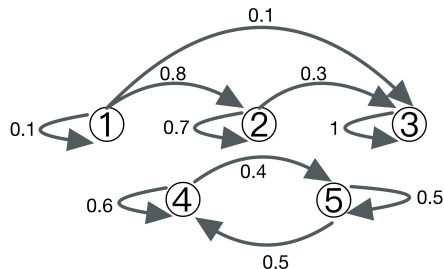
Definition

A state diagram for a Markov chain is a directed graph where the nodes represent the states and the edges represent possible one-step transitions. More precisely, the state diagram contains an edge from node i to node j if and only if $P_{i,j} > 0$.

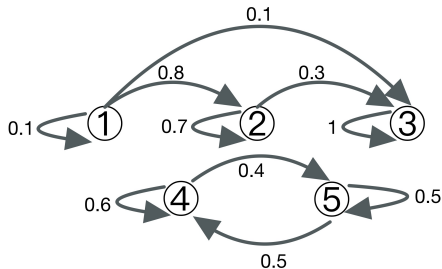
State diagram example

Let $\{X_n\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.8 & 0.1 & 0 & 0 \\ 0 & 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$



State diagram and accessible states



If $i, j \in \mathcal{S}$, then $i \rightarrow j$ if and only if the state diagram contains at least one directed path from i to j .

If such a path exists, we have:

$$P_{i,j}^n \geq P_{i,k_1} \cdot P_{k_1,k_2} \cdot \dots \cdot P_{k_{n-2},k_{n-1}} \cdot P_{k_{n-1},j} > 0,$$

$k_1, \dots, k_{n-1}$ are some states such that $P_{i,k_1}, P_{k_m,k_{m+1}}, P_{k_{n-1},j} > 0, m = 1, \dots, n-2$.

Communicating states

Definition

States i and j communicate, denoted as $i \leftrightarrow j$, if $i \rightarrow j$ and $j \rightarrow i$.

The relation $\leftrightarrow$ is an equivalence relation. That is, $\leftrightarrow$ satisfies the following properties:

- Reflexivity: $i \leftrightarrow i$ (follows since we always have $P_{i,i}^0 = 1 > 0$).
- Symmetry: $i \leftrightarrow j$ if and only if $j \leftrightarrow i$ (follows directly from the definition).
- Transitivity: $i \leftrightarrow j$ and $j \leftrightarrow k$ implies $i \leftrightarrow k$ (requires a proof).

Communicating states: proof of transitivity

To prove transitivity we assume that $i \leftrightarrow j$ and $j \leftrightarrow k$.

Hence, in particular $i \rightarrow j$ and $j \rightarrow k$, implying that there exists $m, n \geq 0$ such that $P_{i,j}^m > 0$ and $P_{j,k}^n > 0$.

By the Chapman-Kolmogorov equations, we have:

$$P_{i,k}^{m+n} = \sum_{r \in S} P_{i,r}^m P_{r,k}^n \geq P_{i,j}^m \cdot P_{j,k}^n > 0.$$

Hence, by definition $i \rightarrow k$.

By a similar argument we can show that $k \rightarrow i$ as well. Hence, we conclude that $i \leftrightarrow k$.

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Lecture 4. First passage probabilities.

Classification of states

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