

UiO : **Department of Mathematics**
University of Oslo

Lecture 5. Classification of states

STK2130 – Modelling av stokastiske
prosesser



Anton Yurchenko-Tytarenko



NIELS HENRIK ABEL
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MATEMATIKER, BERØMT FOR
BANEKRYTENDE ARBEIDER INNEN
LIGNINGSTEORI, UENDELIGE REKKER
OG ELLIPTISKE FUNKSJONER



Accessible states

Let $\{X_n\}$ be a Markov chain with state space $\mathcal{S}$ and transition probability matrix $\mathbf{P}$.

Definition

State j is said to be accessible from state i , denoted as $i \rightarrow j$, if $P_{i,j}^n > 0$ for some $n \geq 0$.

$$i \rightarrow j \iff \mathbb{P} \left(\bigcup_{k=0}^{\infty} \{X_k = j\} \mid X_0 = i \right) > 0.$$

Communicating states

Definition

States i and j communicate, denoted as $i \leftrightarrow j$, if $i \rightarrow j$ and $j \rightarrow i$.

The relation $\leftrightarrow$ is an equivalence relation. That is, $\leftrightarrow$ satisfies the following properties:

- Reflexivity: $i \leftrightarrow i$ (follows since we always have $P_{i,i}^0 = 1 > 0$).
- Symmetry: $i \leftrightarrow j$ if and only if $j \leftrightarrow i$ (follows directly from the definition).
- Transitivity: $i \leftrightarrow j$ and $j \leftrightarrow k$ implies $i \leftrightarrow k$ (requires a proof).

Communicating states

Definition

Two states that communicate are said to be *in the same (equivalence) class*.

Proposition

Two classes of states are either identical or disjoint.

PROOF. Assume that $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ represent two equivalence classes, and assume that $\mathcal{A} \cap \mathcal{B} \neq \emptyset$. That is, there exists a state i such that $i \in \mathcal{A} \cap \mathcal{B}$.

Then choose $j \in \mathcal{A}$ and $k \in \mathcal{B}$ arbitrarily.

Now, $i, j \in \mathcal{A}$ implies that $i \leftrightarrow j$ and $i, k \in \mathcal{B}$ implies that $i \leftrightarrow k$.

Hence, by transitivity, we also have $j \leftrightarrow k$. That is, j and k belong to the same equivalence class. Since this holds for any $j \in \mathcal{A}$ and $k \in \mathcal{B}$, this implies that $\mathcal{A} = \mathcal{B}$.

Irreducible Markov chains

Definition

The equivalence classes partition the state space $\mathcal{S}$ into a number of disjoint sets. A Markov chain is called *irreducible* if the number of equivalence classes is one.

Example 4.15

Consider a Markov chain with state space $S = \{0, 1, 2\}$ and transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

We then observe:

- Since $P_{0,1} = \frac{1}{2} > 0$, it follows that $0 \rightarrow 1$.
- Since $P_{1,0} = \frac{1}{2} > 0$, it follows that $1 \rightarrow 0$.
- Since $P_{1,2} = \frac{1}{4} > 0$, it follows that $1 \rightarrow 2$.
- Since $P_{2,1} = \frac{1}{3} > 0$, it follows that $2 \rightarrow 1$.

Hence, $0 \leftrightarrow 1$ and $1 \leftrightarrow 2$, and, by transitivity, $0 \leftrightarrow 2$ as well. Thus, the Markov chain is irreducible.

Example 4.15

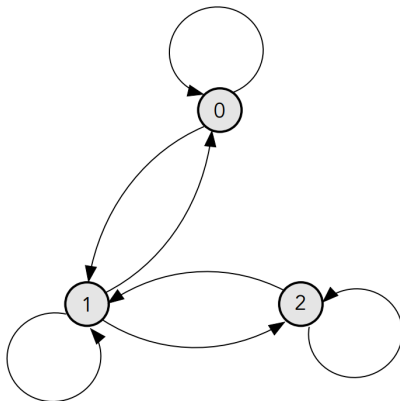


Figure: State diagram of an irreducible Markov chain with one class $\{0, 1, 2\}$

Example 4.16

A Markov chain with state space $S = \{0, 1, 2, 3\}$ and matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_{0,1} = P_{1,0} = \frac{1}{2} \implies 0 \leftrightarrow 1,$$

$$P_{0,j} = P_{1,j} = 0 \implies 0, 1 \not\leftrightarrow j, \quad j = 2, 3,$$

$$P_{2,i} = \frac{1}{4} \implies 2 \rightarrow i, \quad i = 0, 1, 2, 3,$$

$$P_{3,i} = 0 \implies 3 \not\leftrightarrow i, \quad i = 0, 1, 2.$$

The Markov chain has classes $\{0, 1\}$, $\{2\}$ and $\{3\}$, and is *not* irreducible.

Example 4.16

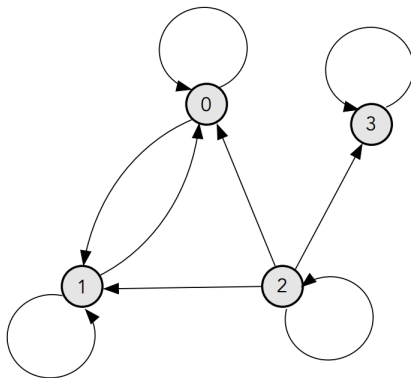


Figure: State diagram of a Markov chain with three classes $\{0, 1\}$, $\{2\}$ and $\{3\}$.

Recurrent and transient states

We consider the probabilities:

$$f_i := \mathbb{P} \left(\bigcup_{r=1}^{\infty} \{X_r = i\} \mid X_0 = i \right), \quad i \in \mathcal{S}.$$

Definition

- State i is *recurrent* if $f_i = 1$.
- State i is *transient* if $f_i < 1$.

Assume that $X_0 = i$, and let N_i denote the number of times $X_n = i$.

- If i is *recurrent*, then $\mathbb{P}(N_i = \infty \mid X_0 = i) = 1$.
- If i is *transient*, then $\mathbb{P}(N_i = n \mid X_0 = i) = f_i^{n-1}(1 - f_i)$, $n = 1, 2, \dots$

If i is transient and $X_0 = i$, then $N_i \mid X_0 = i$ has a geometric distribution with

$$\mathbb{E}[N_i \mid X_0 = i] = 1/(1 - f_i) < \infty.$$

Proposition 4.1

Proposition

- State i is recurrent, if $\sum_{n=1}^{\infty} P_{i,i}^n = \infty$.
- State i is transient, if $\sum_{n=1}^{\infty} P_{i,i}^n < \infty$.

PROOF. Let $I_i^{(n)} = \mathbb{1}(X_n = i)$, $n = 0, 1, \dots$. We can then write:

$$N_i = \sum_{n=0}^{\infty} I_i^{(n)}.$$

Hence, we have:

$$\mathbb{E}[N_i \mid X_0 = i] = \sum_{n=0}^{\infty} \mathbb{E}[I_i^{(n)} \mid X_0 = i] = \sum_{n=0}^{\infty} \mathbb{P}(X_n = i \mid X_0 = i) = \sum_{n=0}^{\infty} P_{i,i}^n.$$

Corollary 4.2

Corollary

If state i is recurrent, and $i \leftrightarrow j$, then state j is recurrent as well. Thus, recurrence is a class property.

PROOF. Since $i \leftrightarrow j$, there exists k and m such that $P_{i,j}^k > 0$ and $P_{j,i}^m > 0$.

Hence, for any $n = 1, 2, \dots$ we have:

$$P_{j,j}^{m+n+k} \geq P_{j,i}^m \cdot P_{i,i}^n \cdot P_{i,j}^k.$$

Summing over all n , and using that i is recurrent, $P_{i,j}^k > 0$ and $P_{j,i}^m > 0$ we get:

$$\sum_{n=1}^{\infty} P_{j,j}^{m+n+k} \geq P_{j,i}^m \cdot P_{i,j}^k \cdot \sum_{n=1}^{\infty} P_{i,i}^n = \infty.$$

Hence, we conclude that j is recurrent as well.



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