

Lecture 6. Long-Run Proportions and Limiting Probabilities

STK2130 – Modelling av stokastiske prosesser



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Recurrent and transient states

Let $\{X_n\}$ be a Markov chain with state space S and transition probability matrix $\mathbf{P}$.

In the last lecture we considered the probabilities:

$$f_i = \mathbb{P} \left(\bigcup_{r=1}^{\infty} \{X_r = i\} \mid X_0 = i \right), \quad i \in S.$$

We then introduced the following concepts:

- State i is *recurrent* if $f_i = 1$.
- State i is *transient* if $f_i < 1$.

Proposition 4.1

- State i is recurrent, if $\sum_{n=1}^{\infty} P_{i,i}^n = \infty$.
- State i is transient, if $\sum_{n=1}^{\infty} P_{i,i}^n < \infty$.

Corollary 4.2

If state i is recurrent, and $i \leftrightarrow j$, then state j is recurrent as well. Thus, recurrence is a class property.

More corollaries

Assume that $X_0 = i$, and let N_i denote the number of times $X_n = i$.

- If i is *recurrent*, then $\mathbb{P}(N_i = \infty \mid X_0 = i) = 1 \implies \mathbb{E}[N_i \mid X_0 = i] = \infty$.
- If i is *transient*, then $\mathbb{P}(N_i = n \mid X_0 = i) = f_i^{n-1}(1 - f_i) \implies \mathbb{E}[N_i \mid X_0 = i] = 1/(1 - f_i) < \infty$.

Corollary

- If state i is transient and $i \leftrightarrow j$, then state j must also be transient. For if j were recurrent then, by Corollary 4.2, i would also be recurrent contradicting that i is transient. Thus, *transience is a class property as well*.
- We verified that, if state i is transient, then $\mathbb{E}[N_i \mid X_0 = i] < \infty$. That means that $N_i \mid X_0 = i$ is a *finite* random variable (i.e. $\mathbb{P}(N_i = \infty \mid X_0 = i) = 0$). By definition of N_i , this implies that transient state i can be visited only *finite* number of times.
- Observation above means that, if $\{X_n\}$ is a Markov chain with a *finite* state space, then not all states can be transient. This implies that least one of the states must be *recurrent*.
- If $\{X_n\}$ is *irreducible* as well, then all states are *recurrent*.

Example 4.17

Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It is easy to verify that $i \leftrightarrow j$ for all $i, j \in \mathcal{S}$. Hence, the Markov chain is irreducible and thus all states must be recurrent.

Example 4.17

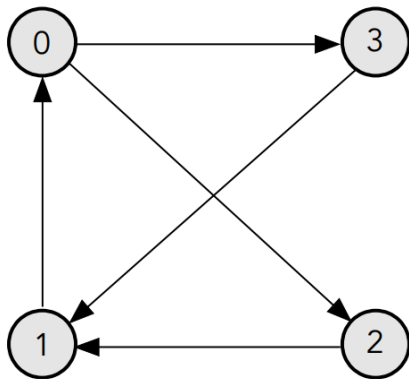


Figure: State diagram of an irreducible Markov chain with one class $\{0, 1, 2, 3\}$

Example 4.18

Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, 4\}$ and transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

This chain has classes $\{0, 1\}$, $\{2, 3\}$ and $\{4\}$. The first two classes are recurrent and the third is transient.

Example 4.18

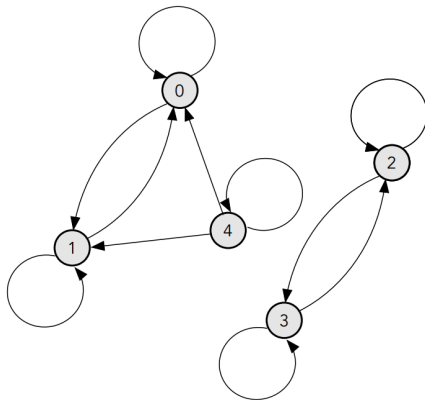


Figure: State diagram of a Markov chain with classes $\{0, 1\}$, $\{2, 3\}$ and $\{4\}$

Example 4.19. Random walk

Consider a Markov chain with state space $\mathcal{S} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and, $0 < p < 1$ and

$$P_{i,i+1} = p, \quad P_{i,i-1} = 1 - p, \quad i \in \mathcal{S}.$$

It is obvious that $i \leftrightarrow j$ for all $i, j \in \mathcal{S}$. Hence, according to Corollary 4.2, all states are either recurrent or transient.

In order to check for recurrence, it is sufficient to check if $\sum_{n=1}^{\infty} P_{0,0}^n = \infty$.

Thus, we assume that $X_0 = 0$, and observe that in this case X_n is odd if n is odd, and X_n is even if n is even. Hence, since 0 is even, we have:

$$P_{0,0}^{2n-1} = 0, \quad n = 1, 2, \dots,$$
$$P_{0,0}^{2n} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!n!} (p(1-p))^n, \quad n = 1, 2, \dots.$$

Example 4.19. Random walk

We then use Stirling's formula for $n!$:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \quad n \rightarrow \infty.$$

From this we get:

$$\frac{(2n)!}{n!n!} \sim \frac{\sqrt{2\pi}e^{-2n}(2n)^{2n+\frac{1}{2}}}{(n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi})^2} = \frac{4^n}{\sqrt{\pi n}}.$$

Hence:

$$P_{0,0}^{2n} = \frac{(2n)!}{n!n!} (p(1-p))^n \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}}, \quad n \rightarrow \infty.$$

Example 4.19. Random walk

This implies that convergence/divergence of the series

$$\sum_{n=1}^{\infty} P_{0,0}^{2n}$$

can be determined via convergence/divergence of the series

$$\sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{\sqrt{\pi n}}.$$

This series is divergent if and only if $p = 1/2$. Hence, the states are recurrent if and only if $p = 1/2$.

Section 4.4. Long-Run Proportions and Limiting Probabilities

Long-Run Proportions and Limiting Probabilities

For pairs of states $i \neq j$ we let $f_{i,j}$ denote the probability that the Markov chain, starting in state i , will ever make a transition into state j :

$$\begin{aligned} f_{i,j} &= \mathbb{P}(X_n = j \text{ for some } n > 0 \mid X_0 = i) \\ &= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{X_n = j\} \mid X_0 = i\right). \end{aligned}$$

We recall that $i \rightarrow j$ if and only if $f_{i,j} > 0$. We now show that:

Proposition 4.3.

If i is recurrent and $i \leftrightarrow j$, then $f_{i,j} = 1$.



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