

Lecture 7. Long-Run Proportions and Limiting Probabilities

STK2130 – Modelling av stokastiske prosesser



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Long-Run Proportions and Limiting Probabilities

For pairs of states $i \neq j$ we let $f_{i,j}$ denote the probability that the Markov chain, starting in state i , will ever make a transition into state j :

$$\begin{aligned} f_{i,j} &= \mathbb{P}(X_n = j \text{ for some } n > 0 \mid X_0 = i) \\ &= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{X_n = j\} \mid X_0 = i\right). \end{aligned}$$

We recall that $i \rightarrow j$ if and only if $f_{i,j} > 0$. We now show that:

Proposition 4.3.

If i is recurrent and $i \leftrightarrow j$, then $f_{i,j} = 1$.

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If i is recurrent and $i \leftrightarrow j$, then $f_{i,j} = 1$.

PROOF. Let $X_0 = i$. Since the state i is recurrent, Markov chain $\{X_n\}$ will return to i infinitely many times, i.e. *with probability 1* there exists an infinite sequence of (random!) times $0 = k_0 < k_1 < k_2 < \dots$, such that $X_{k_r} = i, r = 0, 1, 2, \dots$

Since $i \leftrightarrow j$, there exists some $n > 0$ such that $P_{i,j}^n > 0$.

Let us say that we *had a successful hit on step r* if

$$X_{k_0+n} \neq j, \quad \dots \quad X_{k_{r-1}+n} \neq j, \quad X_{k_r+n} = j.$$

and consider a random variable

$$Z := \min\{r \geq 0 : X_{k_r+n} = j\}$$

– the number of steps before the first successful hit.

Proposition 4.3

Let us say that we *had a successful hit on step r* if

$$X_{k_0+n} \neq j, \quad \dots \quad X_{k_{r-1}+n} \neq j, \quad X_{k_r+n} = j.$$

In other words:

- we start from $X_0 = i$ and check whether $X_n = j$ — if yes, we say that we have a successful hit on step 0 (it happens with probability $P_{i,j}^n$);
- if on step 0 there were no hit, we take $X_{k_1} = i$ and check whether $X_{k_1+n} = j$ — if yes, we say that we have a successful hit on step 1 (it happens with probability $(1 - P_{i,j}^n)P_{i,j}^n$);
- if on step 1 there were no hit either, we take $X_{k_2} = i$ and check whether $X_{k_2+n} = j$ — if yes, we say that we have a successful hit on step 2 (it happens with probability $(1 - P_{i,j}^n)^2 P_{i,j}^n$);
- ...

We then introduce:

$$Z := \min\{r \geq 0 : X_{k_r+n} = j\}$$

– the number of steps before the first successful hit.

We have just seen that:

$$\mathbb{P}(Z = z) = P_{i,j}^n \cdot (1 - P_{i,j}^n)^z, \quad z = 0, 1, 2, \dots$$

Proposition 4.3

This means that Z — the number of steps before the first successful hit — has a geometric distribution with parameter $P_{i,j}^n > 0$. In particular, Z is finite with probability 1, and therefore the first hit will eventually occur.

More precisely,

$$1 \geq f_{i,j} = \mathbb{P} \left(\bigcup_{n=1}^{\infty} \{X_n = j\} \mid X_0 = i \right) \geq \sum_{z=0}^{\infty} \mathbb{P}(Z = z) = 1.$$

Hence, we conclude $f_{i,j} = 1$.

Remark

- In the proof, $i \leftrightarrow j$ ensures that an attempt to reach j from i can be successful with positive probability.
- Recurrence of i ensures that we will have an infinite number of attempts “to hit” state j from state i .

Positive and null recurrency

Assume that j is a recurrent state and introduce

$$N_j := \min\{n > 0 : X_n = j\}$$

– the first time of getting to state j (does not include $n = 0$).

Warning!

Do not confuse this N_j (the first moment of reaching j) with N_i from before (it was occasionally used to denote the number of times $\{X_n\}$ visited i). Until further notice,

$$N_j := \min\{n > 0 : X_n = j\}.$$

We then let

$$m_j := \mathbb{E}[N_j \mid X_0 = j].$$

That is, m_j is the expected number of steps until the Markov chain returns to state j given that it starts out in state j .

Positive and null recurrence

We have that j – recurrent,

$$N_j := \min\{n > 0 : X_n = j\}, \quad m_j := \mathbb{E}[N_j \mid X_0 = j].$$

Remark

Since j is recurrent, we know that $\mathbb{P}(N_j < \infty \mid X_0 = j) = 1$. Still, depending on the distribution of N_j , it may happen that $\mathbb{E}[N_j \mid X_0 = j] = \infty$.

Definition

- If $m_j < \infty$, we say that j is *positive* recurrent.
- If $m_j = \infty$, we say that j is *null* recurrent.

Positive and null recurrency

Let π_j be the long-run proportion of time the Markov chain is in state j (if exists!):

$$\pi_j = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \mathbb{1}(X_r = j).$$

Proposition 4.4

If the Markov chain is irreducible and recurrent, then π_j exist, do not depend on the initial state X_0 and:

$$\pi_j = 1/m_j.$$

Remark

If $m_j = \infty$, then $\pi_j = 0$.

Proof of Proposition 4.4

Assume that $X_0 = i$, and introduce:

$$T_0 := \min\{r > 0 : X_r = j\},$$

$$T_1 := \min\{r > 0 : X_{T_0+r} = j\},$$

$$T_k := \min\{r > 0 : X_{T_0+\dots+T_{k-1}+r} = j\}, \quad k = 2, 3, \dots$$

We then note:

- $\mathbb{P}(T_0 < \infty) = f_{i,j} = 1$ (Proposition 4.3 tells us that transition to j will eventually occur).
- $T_1, T_2, \dots$ are independent and identically distributed (follows from the Markov property, can be formally checked by conditioning).
- $\mathbb{E}[T_k] = m_j, k = 1, 2, \dots$ (each T_k is essentially the number of steps until the Markov chain returns to state j given that it starts out in state j , and m_j is by definition the expectation of *exactly that*).

Hence, by the *strong law of large numbers*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n T_k = m_j$$

with probability 1.

Proof of Proposition 4.4

$T_0 + \sum_{k=1}^n T_k$ is the time the chain enters state j for the $(n+1)$ -st time.

The proportion of time the chain has been in state j at this point of time is:

$$\frac{\text{Number of times in } j}{\text{Total time}} = \frac{n+1}{T_0 + \sum_{k=1}^n T_k}.$$

Hence, the long-run proportion is given by:

$$\pi_j = \lim_{n \rightarrow \infty} \frac{n+1}{T_0 + \sum_{k=1}^n T_k} = \lim_{n \rightarrow \infty} \frac{1}{\frac{T_0}{n+1} + \frac{n}{n+1} \frac{1}{n} \sum_{k=1}^n T_k} = \frac{1}{m_j}.$$

Remark

We have that $m_j < \infty$ if and only if $1/m_j > 0$. Thus, state j is positive recurrent if and only if $\pi_j = 1/m_j > 0$.

Positive recurrence as a “class property”

Proposition 4.5

Let $\{X_n\}$ be an irreducible Markov chain. If state i is positive recurrent, then *all* states are positive recurrent as well.

SKETCH OF PROOF. Since i is *positive* recurrent, it is also recurrent. Therefore, since recurrence is a class property (and we have only one class), then *all* states are recurrent. Moreover, by Proposition 4.4, we know that long-run proportions π_j exist for all states j . Furthermore, by Remark after Proposition 4.4, $\pi_i > 0$ as i is positive recurrent.

We need to check that for any state j : $\pi_j > 0$.

Positive recurrence as a “class property”

We will need the following *magical theorem*.

Dominated convergence theorem

Let $\xi_1, \dots, \xi_n, \dots$ be a sequence of random variables such that

- with probability 1, there exists a limit $\lim_{n \rightarrow \infty} \xi_n =: \xi$
- there exists a random variable η with $\mathbb{E}\eta < \infty$ such that for all $n \geq 1$:

$$|\xi_n| \leq \eta \quad \text{with probability 1.}$$

Then $\mathbb{E}[\xi] < \infty$ and

$$\lim_{n \rightarrow \infty} \mathbb{E}[\xi_n] = \mathbb{E}[\xi].$$

What is the deal?

In general, it is **not** true that

$$\lim_{n \rightarrow \infty} \xi_n = \xi \text{ with probability 1} \implies \lim_{n \rightarrow \infty} \mathbb{E}[\xi_n] = \mathbb{E}[\xi].$$

Positive recurrence as a “class property”

Consider an arbitrary state j . Since $i \leftrightarrow j$, there exists an $n > 0$ such that $P_{i,j}^n > 0$. Moreover, since $\pi_j = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \mathbb{1}(X_r = j)$, we can write using dominated convergence theorem that

$$\begin{aligned}\pi_j &= \mathbb{E} \left[\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \mathbb{1}(X_r = j) \right] = \lim_{m \rightarrow \infty} \mathbb{E} \left[\frac{1}{m} \sum_{r=1}^m \mathbb{1}(X_r = j) \right] = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \mathbb{E} [\mathbb{1}(X_r = j)] \\&= \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \mathbb{P}(X_r = j) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^n \mathbb{P}(X_r = j) + \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \mathbb{P}(X_r = j) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \mathbb{P}(X_r = j) \\&= \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \sum_{s \in \mathcal{S}} \mathbb{P}(X_r = j \mid X_{r-n} = s) \mathbb{P}(X_{r-n} = s) \geq \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \mathbb{P}(X_r = j \mid X_{r-n} = i) \mathbb{P}(X_{r-n} = i) \\&= \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m P_{i,j}^n \mathbb{P}(X_{r-n} = i) = P_{i,j}^n \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \mathbb{P}(X_{r-n} = i) = P_{i,j}^n \lim_{m \rightarrow \infty} \frac{m-n}{m} \frac{1}{m-n} \sum_{k=1}^{m-n} \mathbb{P}(X_k = i) \\&= P_{i,j}^n \lim_{m \rightarrow \infty} \frac{1}{m-n} \sum_{k=1}^{m-n} \mathbb{P}(X_k = i) = P_{i,j}^n \pi_i > 0.\end{aligned}$$

Therefore $\pi_j > P_{i,j}^n \pi_i > 0$ and we get that j is positive recurrent.

Positive recurrence is a class property

Corollary 4.5.1

Let $\{X_n\}$ be an irreducible Markov chain. If state i is null recurrent, then *all* states are null recurrent as well.

PROOF. Assume that i is null recurrent and j is some other state. Since i is recurrent, j is also recurrent, so there are only two options: either j is null recurrent, or j is positive recurrent.

If j is positive recurrent, Proposition 4.5 implies that i is positive recurrent as well. However, this contradicts the assumption.

Positive recurrence is a class property

Corollary 4.5.2

An irreducible finite state Markov chain must be positive recurrent.

PROOF. We know that from the beginning of the lecture that irreducible finite state Markov chain must be recurrent (i.e. we can talk about positive or null recurrence in the first place).

By Proposition 4.5, we have only two (mutually excluding!) options:

- all states in an irreducible Markov chain are positive recurrent;
- all states in an irreducible Markov chain are null recurrent.

However, for a finite state Markov chain, second option is not possible. Indeed, if all states are null recurrent, then $\pi_i = 0$ for all $i \in \mathcal{S}$. In other words, *for all* $i \in \mathcal{S}$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \mathbb{1}(X_r = i) = 0.$$

This would imply that

$$0 = \sum_{i \in \mathcal{S}} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \mathbb{1}(X_r = i) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \sum_{i \in \mathcal{S}} \mathbb{1}(X_r = i) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n 1 = 1.$$



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