

Lecture 8. Limiting probabilities

STK2130 – Modelling av stokastiske prosesser



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MATEMATIKER, BERØMT FOR
BANEBRYTENDE ARBEIDER INNEN
LIGNINGSTEORI, UENDELIGE REKKER
OG ELLIPTISKE FUNKSJONER



Brush up

Assume that j is a recurrent state and introduce

$$N_j := \min\{n > 0 : X_n = j\}, \quad m_j := \mathbb{E}[N_j \mid X_0 = j].$$

Definition

- If $m_j < \infty$, we say that j is *positive* recurrent.
- If $m_j = \infty$, we say that j is *null* recurrent.

Proposition 4.4

If the Markov chain is irreducible and recurrent, then π_j exist, do not depend on the initial state X_0 and:

$$\pi_j = 1/m_j.$$

Positive recurrence as a “class property”

Proposition 4.5

Let $\{X_n\}$ be an irreducible Markov chain. If state i is positive recurrent, then *all* states are positive recurrent as well.

SKETCH OF PROOF. Since i is *positive* recurrent, it is also recurrent. Therefore, since recurrence is a class property (and we have only one class), then *all* states are recurrent. Moreover, by Proposition 4.4, we know that long-run proportions π_j exist for all states j . Furthermore, by Remark after Proposition 4.4, $\pi_i > 0$ as i is positive recurrent.

We need to check that for any state j : $\pi_j > 0$.

Positive recurrence as a “class property”

We will need the following *magical theorem*.

Dominated convergence theorem

Let $\xi_1, \dots, \xi_n, \dots$ be a sequence of random variables such that

- with probability 1, there exists a limit $\lim_{n \rightarrow \infty} \xi_n =: \xi$
- there exists a random variable η with $\mathbb{E}\eta < \infty$ such that for all $n \geq 1$:

$$|\xi_n| \leq \eta \quad \text{with probability 1.}$$

Then $\mathbb{E}[\xi] < \infty$ and

$$\lim_{n \rightarrow \infty} \mathbb{E}[\xi_n] = \mathbb{E}[\xi].$$

What is the deal?

In general, it is **not** true that

$$\lim_{n \rightarrow \infty} \xi_n = \xi \text{ with probability 1} \implies \lim_{n \rightarrow \infty} \mathbb{E}[\xi_n] = \mathbb{E}[\xi].$$

Positive recurrence as a “class property”

Consider an arbitrary state j . Since $i \leftrightarrow j$, there exists an $n > 0$ such that $P_{i,j}^n > 0$. Moreover, since $\pi_j = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \mathbb{1}(X_r = j)$, we can write using dominated convergence theorem that

$$\begin{aligned}\pi_j &= \mathbb{E} \left[\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \mathbb{1}(X_r = j) \right] = \lim_{m \rightarrow \infty} \mathbb{E} \left[\frac{1}{m} \sum_{r=1}^m \mathbb{1}(X_r = j) \right] = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \mathbb{E} [\mathbb{1}(X_r = j)] \\&= \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \mathbb{P}(X_r = j) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=1}^n \mathbb{P}(X_r = j) + \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \mathbb{P}(X_r = j) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \mathbb{P}(X_r = j) \\&= \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \sum_{s \in \mathcal{S}} \mathbb{P}(X_r = j \mid X_{r-n} = s) \mathbb{P}(X_{r-n} = s) \geq \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \mathbb{P}(X_r = j \mid X_{r-n} = i) \mathbb{P}(X_{r-n} = i) \\&= \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m P_{i,j}^n \mathbb{P}(X_{r-n} = i) = P_{i,j}^n \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{r=n+1}^m \mathbb{P}(X_{r-n} = i) = P_{i,j}^n \lim_{m \rightarrow \infty} \frac{m-n}{m} \frac{1}{m-n} \sum_{k=1}^{m-n} \mathbb{P}(X_k = i) \\&= P_{i,j}^n \lim_{m \rightarrow \infty} \frac{1}{m-n} \sum_{k=1}^{m-n} \mathbb{P}(X_k = i) = P_{i,j}^n \pi_i > 0.\end{aligned}$$

Therefore $\pi_j > P_{i,j}^n \pi_i > 0$ and we get that j is positive recurrent.

Positive recurrence is a class property

Corollary 4.5.1

Let $\{X_n\}$ be an irreducible Markov chain. If state i is null recurrent, then *all* states are null recurrent as well.

PROOF. Assume that i is null recurrent and j is some other state. Since i is recurrent, j is also recurrent, so there are only two options: either j is null recurrent, or j is positive recurrent.

If j is positive recurrent, Proposition 4.5 implies that i is positive recurrent as well. However, this contradicts the assumption.

Positive recurrence is a class property

Corollary 4.5.2

An irreducible finite state Markov chain must be positive recurrent.

PROOF. We know that from the beginning of the lecture that irreducible finite state Markov chain must be recurrent (i.e. we can talk about positive or null recurrence in the first place).

By Proposition 4.5, we have only two (mutually excluding!) options:

- all states in an irreducible Markov chain are positive recurrent;
- all states in an irreducible Markov chain are null recurrent.

However, for a finite state Markov chain, second option is not possible. Indeed, if all states are null recurrent, then $\pi_i = 0$ for all $i \in \mathcal{S}$. In other words, *for all* $i \in \mathcal{S}$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \mathbb{1}(X_r = i) = 0.$$

This would imply that

$$0 = \sum_{i \in \mathcal{S}} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \mathbb{1}(X_r = i) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \sum_{i \in \mathcal{S}} \mathbb{1}(X_r = i) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n 1 = 1.$$

Long-run proportion of states (heuristic argument)

We have that:

$$\begin{aligned} & \text{Long-run proportion of transitions that go from } i \text{ to } j \\ &= \text{Long run proportion of being in state } i \times \text{probability of transition from } i \text{ to } j \\ &= \pi_i P_{i,j}. \end{aligned}$$

Hence, by summing over all possible preceding states of j , we get:

$$\pi_j = \sum_{i \in S} \pi_i P_{i,j}.$$

Long-run proportion of states (without proof)

Theorem 4.1

Consider an irreducible Markov chain. If the chain is positive recurrent, then the long-run proportions are the unique solution of the equations:

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i P_{i,j}, \quad \text{for all } j \in \mathcal{S},$$

$$\sum_{j \in \mathcal{S}} \pi_j = 1.$$

Moreover, if there is no solution of these linear equations, then the Markov chain is either transient or null recurrent, and $\pi_j = 0$ for all $j \in \mathcal{S}$.

Symmetric random walk

Consider a Markov chain with state space $\mathcal{S} = \{\dots, -2, -1, 0, 1, 2, \dots\} = \mathbb{Z}$ and where:

$$P_{i,i+1} = P_{i,i-1} = \frac{1}{2}, \quad i \in \mathcal{S}.$$

By Example 4.19 we know that this chain is irreducible and recurrent. Therefore, by Proposition 4.4, there exist long-run proportions π_i , $i \in \mathcal{S}$ that do not depend on the initial state X_0 .

Moreover, by Theorem 4.1 we have that:

$$\pi_i = \pi_{i-1} \cdot \frac{1}{2} + \pi_{i+1} \cdot \frac{1}{2}.$$

We now assume that $X_0 = i$, where $i \in \mathcal{S}$ is arbitrarily chosen. Then it follows by symmetry that $\pi_{i-1} = \pi_{i+1}$, and hence

$$\pi_{i-1} = \pi_i = \pi_{i+1}.$$

Symmetric random walk

Similarly, it follows by Theorem 4.1 that:

$$\pi_{i+1} = \pi_i \cdot \frac{1}{2} + \pi_{i+2} \cdot \frac{1}{2},$$

$$\pi_{i-1} = \pi_i \cdot \frac{1}{2} + \pi_{i-2} \cdot \frac{1}{2}.$$

Since $\pi_{i-1} = \pi_i = \pi_{i+1}$, this implies that

$$\pi_{i-2} = \pi_i = \pi_{i+2}.$$

Continuing in the same way, we get that

$$\pi_{i-k} = \pi_i = \pi_{i+k}, \quad k = 1, 2, \dots$$

Since the initial state i was arbitrarily chosen, we conclude that the long-run proportions are the same for all states regardless of the initial state, and denote this common proportion by π .

Symmetric random walk

If the chain is positive recurrent, it follows by Theorem 4.1 that:

$$\sum_{j \in \mathcal{S}} \pi_j = \pi \cdot \sum_{j \in \mathcal{S}} 1 = 1.$$

However, $\sum_{j \in \mathcal{S}} 1 = \infty$, so this would imply that $\pi = 0$. Thus, we conclude that *the chain is null recurrent*.

Long-Run Proportions and Limiting Probabilities

Previously (in Lecture 2) we introduced the probabilities:

$$\pi_i^{(n)} = \mathbb{P}(X_n = i), \quad i \in \mathcal{S}, \quad n = 0, 1, 2, \dots$$

Moreover, we defined:

$$\lim_{n \rightarrow \infty} \pi_i^{(n)} = \pi_i, \quad i \in \mathcal{S},$$

and showed that (*if it turns out that the limit above exists*), for a finite state space,

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i P_{i,j}, \quad j \in \mathcal{S}, \quad \sum_{j \in \mathcal{S}} \pi_j = 1.$$

Assuming that all assumptions of Theorem 4.1 hold (in particular, if there exists a unique solution to these equations) and the limit above exists, it follows by Theorem 4.1 that these limiting probabilities are the same as the long-run proportions.

Remark

Note that currently we do not know *when* the limits above exist.

Long-Run Proportions and Limiting Probabilities

Remark

The set of equations

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i P_{i,j}, \quad j \in \mathcal{S},$$

may be written in matrix form as:

$$\boldsymbol{\pi} = \boldsymbol{\pi} \cdot \mathbf{P},$$

where:

$\boldsymbol{\pi}$ = The row vector of long-run proportions = The row vector of limiting probabilities

Example 4.22

$$\mathbb{P}(\text{Rain tomorrow} \mid \text{Rain today}) = \alpha = 0.7,$$

$$\mathbb{P}(\text{Rain tomorrow} \mid \text{No rain today}) = \beta = 0.4,$$

$$\mathbf{P} = \begin{pmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{pmatrix}.$$

In order to find the long-run proportion of rain (π_0) and not-rain (π_1), we solve the equations:

$$\begin{cases} \pi_0 &= \alpha\pi_0 + \beta\pi_1, \\ \pi_1 &= (1 - \alpha)\pi_0 + (1 - \beta)\pi_1, \\ 1 &= \pi_0 + \pi_1. \end{cases}$$

The last equation implies that $\pi_1 = 1 - \pi_0$. By inserting this into the first equation we get:

$$\pi_0 = \alpha\pi_0 + \beta(1 - \pi_0) = (\alpha - \beta)\pi_0 + \beta$$

Example 4.22

We solve the equation:

$$\pi_0 = (\alpha - \beta)\pi_0 + \beta$$

with respect to π_0 , and get that

$$\pi_0 = \frac{\beta}{1 - \alpha + \beta} = \frac{0.4}{1 - 0.7 + 0.4} = \frac{4}{7}.$$

By using the last equation again we find that:

$$\pi_1 = 1 - \pi_0 = \frac{1 - \alpha}{1 - \alpha + \beta} = \frac{1 - 0.7}{1 - 0.7 + 0.4} = \frac{3}{7}.$$

Example 4.23. Mood of an individual

0 = cheerful, 1 = so-so, 2 = glum.

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

In order to find the long-run proportions π_0 , π_1 and π_2 , we solve the equations:

$$\begin{cases} \pi_0 &= 0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2, \\ \pi_1 &= 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2, \\ \pi_2 &= 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2, \\ 1 &= \pi_0 + \pi_1 + \pi_2. \end{cases}$$

SOLUTION:

$$\pi_0 = \frac{21}{62} = 0.3387, \quad \pi_1 = \frac{23}{62} = 0.3710, \quad \pi_2 = \frac{18}{62} = 0.2903.$$

Example 4.23. Mood of an individual

$$\mathbf{P}^{(4)} = \begin{pmatrix} 0.3446 & 0.3734 & 0.2820 \\ 0.3378 & 0.3706 & 0.2916 \\ 0.3330 & 0.3686 & 0.2984 \end{pmatrix},$$

$$\mathbf{P}^{(8)} = \begin{pmatrix} 0.3388 & 0.3710 & 0.2902 \\ 0.3387 & 0.3710 & 0.2903 \\ 0.3386 & 0.3709 & 0.2904 \end{pmatrix},$$

$$\mathbf{P}^{(16)} = \begin{pmatrix} 0.3387 & 0.3710 & 0.2903 \\ 0.3387 & 0.3710 & 0.2903 \\ 0.3387 & 0.3710 & 0.2903 \end{pmatrix},$$

Observation: the solution $\pi_0 = 0.3387$, $\pi_1 = 0.3710$, $\pi_2 = 0.2903$ also corresponds to taking the limit of the n -step transition probability matrix!

Remark

We are *almost* ready to understand why this happens. Explanation of this phenomenon is related to the concept of *ergodicity* and will be considered (with proof) in the upcoming lectures.

Example 4.24. Class mobility

Class mobility problem

A problem of interest to sociologists is to determine the proportion of society that has an upper- or lower-class occupation. One possible mathematical model would be to assume that transitions between social classes of the successive generations in a family can be regarded as transitions of a Markov chain. That is, we assume that the occupation of a child depends only on his or her parent's occupation.

0 = Upper class, 1 = Middle class, 2 = Lower class.

$$\mathbf{P} = \begin{pmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{pmatrix}.$$

In order to find the long-run proportions π_0 , π_1 and π_2 , we solve the equations:

$$\begin{cases} \pi_0 &= 0.45\pi_0 + 0.05\pi_1 + 0.01\pi_2 \\ \pi_1 &= 0.48\pi_0 + 0.70\pi_1 + 0.50\pi_2 \\ \pi_2 &= 0.07\pi_0 + 0.25\pi_1 + 0.49\pi_2 \\ 1 &= \pi_0 + \pi_1 + \pi_2. \end{cases}$$

$$\pi_0 = 0.0624, \quad \pi_1 = 0.6234, \quad \pi_2 = 0.3142.$$

Example 4.24

The solution $\pi_0 = 0.0624$, $\pi_1 = 0.6234$, $\pi_2 = 0.3142$ also corresponds to taking the limit of the n -step transition probability matrix:

$$\mathbf{P}^{(4)} = \begin{pmatrix} 0.0932 & 0.6199 & 0.2869 \\ 0.0623 & 0.6241 & 0.3136 \\ 0.0564 & 0.6229 & 0.3207 \end{pmatrix}$$

$$\mathbf{P}^{(8)} = \begin{pmatrix} 0.0635 & 0.6233 & 0.3132 \\ 0.0624 & 0.6234 & 0.3142 \\ 0.0622 & 0.6235 & 0.3144 \end{pmatrix}$$

$$\mathbf{P}^{(16)} = \begin{pmatrix} 0.0624 & 0.6234 & 0.3142 \\ 0.0624 & 0.6234 & 0.3142 \\ 0.0624 & 0.6234 & 0.3142 \end{pmatrix}$$

Example 4.25. The Hardy-Weinberg Law

Formulation of the problem

Consider *a large population of individuals*, each of whom possesses a particular *pair* of genes, of which each individual gene is classified as being of type A or type a . When two individuals mate, each contributes one of his or her genes, chosen at random, to the resultant offspring.

Assuming that the mating occurs at random, in that each individual is equally likely to mate with any other individual, we are interested in determining the proportions of individuals in the next generation whose genes are AA , aa , or Aa .

Example 4.25. The Hardy-Weinberg Law

Two gene types: A and a

Three possible gene pairs: AA, aa, Aa.

In generation 0 we assume that the proportions of these gene pairs are respectively:

$$p_0 = \text{Proportion of AA}, \quad q_0 = \text{Proportion of aa}, \quad r_0 = \text{Proportion of Aa}.$$

By conditioning on the gene pairs of a parent we get the following probabilities for one of the genes for a given child:

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A \mid AA)p_0 + \mathbb{P}(A \mid aa)q_0 + \mathbb{P}(A \mid Aa)r_0 \\ &= 1 \cdot p_0 + 0 \cdot q_0 + \frac{1}{2} \cdot r_0 = p + \frac{1}{2} \cdot r_0,\end{aligned}$$

$$\begin{aligned}\mathbb{P}(a) &= \mathbb{P}(a \mid AA)p_0 + \mathbb{P}(a \mid aa)q_0 + \mathbb{P}(a \mid Aa)r_0 \\ &= 0 \cdot p_0 + 1 \cdot q_0 + \frac{1}{2} \cdot r_0 = q + \frac{1}{2} \cdot r_0.\end{aligned}$$

Remark

$\mathbb{P}(A)$ and $\mathbb{P}(a)$ can be interpreted as the probability that a randomly chosen gene of a randomly chosen individual in generation 1 is A or a correspondingly.

Example 4.25. The Hardy-Weinberg Law

Hence, the proportions of the gene pairs in generation 1 is:

$$p_1 = \mathbb{P}(AA) = \mathbb{P}(A) \cdot \mathbb{P}(A) = \left(p_0 + \frac{1}{2} \cdot r_0\right)^2$$

$$q_1 = \mathbb{P}(aa) = \mathbb{P}(a) \cdot \mathbb{P}(a) = \left(q_0 + \frac{1}{2} \cdot r_0\right)^2$$

$$r_1 = \mathbb{P}(Aa) = 2\mathbb{P}(A)\mathbb{P}(a) = 2 \cdot \left(p_0 + \frac{1}{2} \cdot r_0\right) \left(q_0 + \frac{1}{2} \cdot r_0\right).$$

Hence, the probabilities of the two gene types for descendants of generation 1 are:

$$\begin{aligned}\mathbb{P}(A) &= p_1 + \frac{1}{2} \cdot r_1 = \left(p_0 + \frac{1}{2} \cdot r_0\right)^2 + \left(p_0 + \frac{1}{2} \cdot r_0\right) \left(q_0 + \frac{1}{2} \cdot r_0\right) \\ &= \left(p_0 + \frac{1}{2} \cdot r_0\right) \left[p_0 + \frac{1}{2} \cdot r_0 + q_0 + \frac{1}{2} \cdot r_0\right] = p_0 + \frac{1}{2} \cdot r_0,\end{aligned}$$

$$\begin{aligned}\mathbb{P}(a) &= q_1 + \frac{1}{2} \cdot r_1 = \left(q_0 + \frac{1}{2} \cdot r_0\right)^2 + \left(p_0 + \frac{1}{2} \cdot r_0\right) \left(q_0 + \frac{1}{2} \cdot r_0\right) \\ &= \left(q_0 + \frac{1}{2} \cdot r_0\right) \left[q_0 + \frac{1}{2} \cdot r_0 + p_0 + \frac{1}{2} \cdot r_0\right] = q_0 + \frac{1}{2} \cdot r_0.\end{aligned}$$

Example 4.25. The Hardy-Weinberg Law

By repeating the same argument we get that the proportions of the gene pairs in generation n is:

$$p_n = \mathbb{P}(AA) = \mathbb{P}(A) \cdot \mathbb{P}(A) = \left(p_0 + \frac{1}{2} \cdot r_0\right)^2$$

$$q_n = \mathbb{P}(aa) = \mathbb{P}(a) \cdot \mathbb{P}(a) = \left(q_0 + \frac{1}{2} \cdot r_0\right)^2$$

$$r_n = \mathbb{P}(Aa) = 2\mathbb{P}(A)\mathbb{P}(a) = 2 \cdot \left(p_0 + \frac{1}{2} \cdot r_0\right) \left(q_0 + \frac{1}{2} \cdot r_0\right)$$

Hence, in generation n the probabilities for the two gene types are:

$$\mathbb{P}(A) = p_0 + \frac{1}{2} \cdot r_0, \quad \mathbb{P}(a) = q_0 + \frac{1}{2} \cdot r_0$$

Remark

Thus, the fractions of the gene pool that are A and a are the same as in the initial generation. From this it follows that, under random mating, in all successive generations after the initial one the percentages of the population having gene pairs AA , aa , and Aa will remain fixed (at the values that we will denote p , q , and r). This is known as the *Hardy–Weinberg law*.

Example 4.25. The Hardy-Weinberg Law

We have seen that for each $n \geq 1$

$$p_n = \left(p_0 + \frac{1}{2} \cdot r_0\right)^2 =: p, \quad q_n = \left(q_0 + \frac{1}{2} \cdot r_0\right)^2 =: q, \quad r_n = 2 \cdot \left(p_0 + \frac{1}{2} \cdot r_0\right) \left(q_0 + \frac{1}{2} \cdot r_0\right) =: r.$$

But also, for any generation $n \geq 1$, the proportion of genes in the next generation is

$$\mathbb{P}(A) = p_0 + \frac{1}{2} \cdot r_0 = p_n + \frac{1}{2} \cdot r_n = p + \frac{1}{2} \cdot r, \quad \mathbb{P}(a) = q_0 + \frac{1}{2} \cdot r_0 = q_n + \frac{1}{2} \cdot r_n = q + \frac{1}{2} \cdot r.$$

That means that

$$p = \left(p + \frac{1}{2} \cdot r\right)^2, \quad q = \left(q + \frac{1}{2} \cdot r\right)^2, \quad r = 2 \cdot \left(p + \frac{1}{2} \cdot r\right) \left(q + \frac{1}{2} \cdot r\right).$$

Example 4.25. The Hardy-Weinberg Law

Suppose now that the gene pair population *has stabilized* in the percentages p, q, r . Let us follow the genetic history of a single individual and her descendants (for simplicity, assume that each individual has exactly one offspring).

We now define:

$X_n =$ The gene pair of an n -th generation child, $n = 1, 2, \dots$,

where the state space is $\mathcal{S} = \{AA, aa, Aa\}$.

Since we have shown that in every generation we have $\mathbb{P}(A) = p + r/2$ and $\mathbb{P}(a) = q + r/2$, the transition matrix for this chain is:

$$\mathbf{P} = \begin{pmatrix} p + r/2 & 0 & q + r/2 \\ 0 & q + r/2 & p + r/2 \\ p/2 + r/4 & q/2 + r/4 & p/2 + q/2 + r/2 \end{pmatrix}$$

It is easy to show that, provided that $p > 0, q > 0$, this Markov chain is irreducible (and thus recurrent).

Example 4.25. The Hardy-Weinberg Law

Remark

We have shown that the distribution p, q, r is stable from generation to generation. This means that

$$p = \mathbb{P}(A) \cdot \mathbb{P}(A) = \left(p + \frac{r}{2}\right)^2$$

$$q = \mathbb{P}(a) \cdot \mathbb{P}(a) = \left(q + \frac{r}{2}\right)^2$$

$$r = 2\mathbb{P}(A)\mathbb{P}(a) = 2 \cdot \left(p + \frac{r}{2}\right) \left(q + \frac{r}{2}\right)$$

We now claim that p, q, r also is the long-time distribution of the Markov chain with transition matrix $\mathbf{P}$. Since obviously $p + q + r = 1$, it is sufficient to verify that

$$(p, q, r) = (p, q, r)\mathbf{P}$$

Example 4.25. The Hardy-Weinberg Law

This follows since we have:

$$p\left(p + \frac{r}{2}\right) + r\left(\frac{p}{2} + \frac{r}{4}\right) = \left(p + \frac{r}{2}\right)^2 = p$$

$$q\left(q + \frac{r}{2}\right) + r\left(\frac{q}{2} + \frac{r}{4}\right) = \left(q + \frac{r}{2}\right)^2 = q,$$

$$\begin{aligned} & p\left(q + \frac{r}{2}\right) + q\left(p + \frac{r}{2}\right) + r\left(\frac{p}{2} + \frac{q}{2} + \frac{r}{2}\right) \\ &= p\left(q + \frac{r}{2}\right) + q\left(p + \frac{r}{2}\right) + \frac{r}{2}\left(p + \frac{r}{2} + q + \frac{r}{2}\right) \\ &= \left(p + \frac{r}{2}\right)\left(q + \frac{r}{2}\right) + \left(q + \frac{r}{2}\right)\left(p + \frac{r}{2}\right) \\ &= 2\left(p + \frac{r}{2}\right)\left(q + \frac{r}{2}\right) = r. \end{aligned}$$



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