

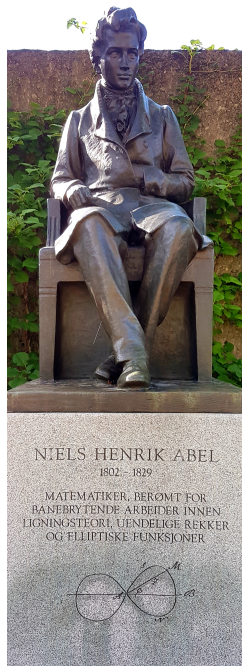
UiO : **Department of Mathematics**
University of Oslo

Lecture 9. Limiting probabilities and ergodicity

STK2130 – Modelling av stokastiske prosesser



Anton Yurchenko-Tytarenko



Stationary probabilities

The long-run proportions $\pi_j, j \in \mathcal{S}$, are called (if they exist) the stationary probabilities of the Markov chain.

In fact, in the setting of Theorem 4.1, if $\mathbb{P}(X_0 = j) = \pi_j, j \in \mathcal{S}$, then $\mathbb{P}(X_n = j) = \pi_j, j \in \mathcal{S}, n = 1, 2, \dots$ as well.

To see this, we let $\pi_j^{(n)} = \mathbb{P}(X_n = j), j \in \mathcal{S}, n = 0, 1, 2, \dots$. Moreover, let $\boldsymbol{\pi}^{(n)}$ denote the vector of $\pi_j^{(n)}, j \in \mathcal{S}$, and let $\boldsymbol{\pi}$ denote the vector of $\pi_j, j \in \mathcal{S}$.

Thus, $\boldsymbol{\pi} = \boldsymbol{\pi}^{(0)}$, and $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$.

By conditioning on X_{n-1} it follows that $\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}^{(n-1)}\mathbf{P}, n = 1, 2, \dots$

Hence, $\boldsymbol{\pi}^{(1)} = \boldsymbol{\pi}^{(0)}\mathbf{P} = \boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$.

By induction this implies that $\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$.

Bounded functions on the state space

Proposition 4.6

Let $\{X_n\}$ be an irreducible Markov chain with stationary probabilities π_j , $j \in \mathcal{S}$, and let f be a bounded function on the state space. Then with probability 1:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(X_n) = \sum_{j \in \mathcal{S}} \pi_j f(j).$$

“Proof”. Let $a_j(N)$ be the amount of time the Markov chain spends in state j during the periods $1, \dots, N$. Then we have:

$$\sum_{n=1}^N f(X_n) = \sum_{j \in \mathcal{S}} a_j(N) f(j).$$

Hence

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(X_n) = \lim_{N \rightarrow \infty} \sum_{j \in \mathcal{S}} \frac{a_j(N)}{N} f(j) = \sum_{j \in \mathcal{S}} \pi_j f(j).$$

Example 4.29. Car insurance

The annual automobile insurance premium depends on the claim history of the client. Each policyholder is given a positive integer-valued state which represents the “reliability” of a client; the annual insurance premium is a function of this state (i.e. the more accidents the client gets into, the more he has to pay as a premium). A policyholder's state changes from year to year in response to the number of claims made by that policyholder in the previous year (see table below).

State	Annual Premium	Next state if			
		0 claims	1 claim	2 claims	3 or more claims
1	200	1	2	3	4
2	250	1	3	4	4
3	400	2	4	4	4
4	600	3	4	4	4

Example 4.29. Car insurance

State space $\mathcal{S} = 1, 2, 3, 4$ bonus classes. We let $f(j)$ denote the premium as a function of state, and assume that:

$$f(1) = 200, \quad f(2) = 250, \quad f(3) = 400, \quad f(4) = 600.$$

Transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0.6065 & 0.3033 & 0.0758 & 0.0144 \\ 0.6065 & 0.0000 & 0.3033 & 0.0902 \\ 0.0000 & 0.6065 & 0.0000 & 0.3935 \\ 0.0000 & 0.0000 & 0.6065 & 0.3935 \end{pmatrix}.$$

The stationary distribution is found by solving $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ combined with the restriction that $\pi_1 + \dots + \pi_4 = 1$, and we get:

$$\pi_1 = 0.3692, \quad \pi_2 = 0.2395, \quad \pi_3 = 0.2103, \quad \pi_4 = 0.1809.$$

Average annual premium is then:

$$f(1) \cdot \pi_1 + f(2) \cdot \pi_2 + f(3) \cdot \pi_3 + f(4) \cdot \pi_4 = 326.375.$$

Limiting Probabilities

We consider a Markov chain with the following transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

From this it follows that:

$$\mathbf{P}^{(4)} = \begin{pmatrix} 0.575 & 0.425 \\ 0.567 & 0.433 \end{pmatrix}$$

$$\mathbf{P}^{(8)} = \begin{pmatrix} 0.571 & 0.429 \\ 0.571 & 0.429 \end{pmatrix}$$

Moreover:

$$\pi_0 = \frac{4}{7} \approx 0.571, \quad \pi_1 = \frac{3}{7} \approx 0.429.$$

Limiting Probabilities

From this example (and other examples we considered before) it is tempting to claim that:

$$\lim_{n \rightarrow \infty} P_{i,j}^n = \pi_j, \quad \text{for all } i, j \in \mathcal{S}.$$

COUNTER EXAMPLE:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we solve the equations $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ and $\pi_0 + \pi_1 = 1$, we get: $\pi_0 = \pi_1 = \frac{1}{2}$.

Limiting Probabilities

In this case we have for $n = 1, 2, \dots$:

$$\mathbf{P}^{(2n)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{P}^{(2n+1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, $\lim_{n \rightarrow \infty} P_{i,j}^{(n)}$ does not exist!

Definition

If a Markov chain can only return to a state in a multiple of $d > 1$ steps, it is said to be *periodic*. A Markov chain which is not periodic is said to be *aperiodic*. An irreducible, positive recurrent, aperiodic Markov chain is said to be *ergodic*.

Limiting Probabilities

Theorem

If a Markov chain with state space S is ergodic (i.e. irreducible + positive recurrent + aperiodic), then the limiting probabilities will always exist, and do not depend on the initial state, and we have:

$$\lim_{n \rightarrow \infty} P_{i,j}^n = \pi_j, \quad \text{for all } i, j \in S.$$

Proof: We only prove the last part of the Theorem. That is, we assume that the limiting probabilities exists and that these probabilities do not depend on the initial state. We may then define:

$$\alpha_j = \lim_{n \rightarrow \infty} P_{i,j}^n, \quad j \in S.$$

By the Chapman-Kolmogorov equations we have:

$$P_{i,j}^{n+1} = \sum_{k \in S} P_{i,k}^n P_{k,j}.$$

Limiting Probabilities

Moreover, we also have that:

$$\sum_{j \in S} P_{i,j}^n = 1.$$

By letting n go to infinity, we then obtain:

$$\alpha_j = \lim_{n \rightarrow \infty} P_{i,j}^{n+1} = \lim_{n \rightarrow \infty} \sum_{k \in S} P_{i,k}^n P_{k,j} = \sum_{k \in S} \alpha_k P_{k,j},$$

$$\lim_{n \rightarrow \infty} \sum_{j \in S} P_{i,j}^n = \sum_{j \in S} \alpha_j = 1.$$

By Theorem 4.1 these equations have a unique solution, and thus we conclude that:

$$\alpha_j = \lim_{n \rightarrow \infty} P_{i,j}^n = \pi_j, \quad j \in S.$$

Periodic Markov chains

We recall that a Markov chain $\{X_n\}$ is said to be periodic if it can only return to a state in a multiple of $d > 1$ steps.

EXAMPLE: Assume that $\{X_n\}$ has state space $\mathcal{S} = \{0, 1\}$, and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Assuming that $X_0 = 0$, it follows that:

$$X_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Thus, this chain can return to a state (0 or 1) in a multiple of 2 steps.

Question

Does periodicity only occur when the chain is deterministic?

Periodic Markov chains

EXAMPLE 1. Assume that $\{X_n\}$ has state space $\mathcal{S} = \{0, 1, 2\}$, and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0.0 & 1.0 & 0.0 \\ 0.5 & 0.0 & 0.5 \\ 0.0 & 1.0 & 0.0 \end{pmatrix}$$

Assuming that $X_0 = 1$, the chain will return to this state for $n = 2, 4, 6, \dots$. Thus, the chain is periodic but not deterministic.

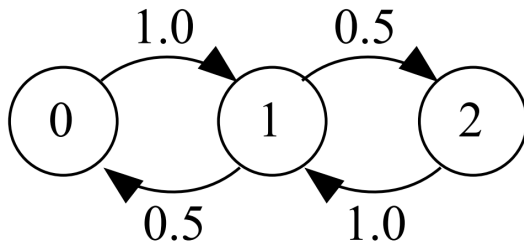


Figure: A non-deterministic periodic Markov chain

Periodic Markov chains

EXAMPLE 2. *One-dimensional random walk.* If $X_0 = 0$, then X_n is even if n is even, and odd if n is odd. The chain can only return to state 0 in an even number of steps. Thus, this chain is periodic but not deterministic.

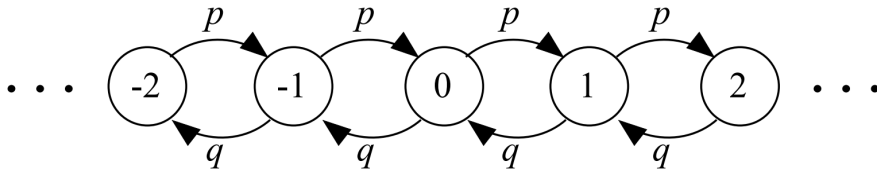


Figure: A one-dimensional random walk

Periodic Markov chains

EXAMPLE 3. Assume that $\{X_n\}$ has state space $S = \{0, 1, 2, 3, 4\}$, and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \end{pmatrix}$$

Assuming that $X_0 = 2$, the chain will return to this state for $n = 3, 6, 9, \dots$. Thus, the chain is periodic but not deterministic.

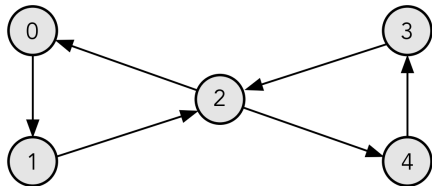


Figure: A non-deterministic periodic Markov chain

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