

Exponential random variables (S2)

Compute: $a > 0$, want to compute

$$I_a = \int_0^{\infty} e^{-ax} dx = \lim_{c \rightarrow \infty} \int_0^c e^{-ax} dx = \lim_{c \rightarrow \infty} \left[\frac{e^{-ax}}{-a} \right]_0^c$$

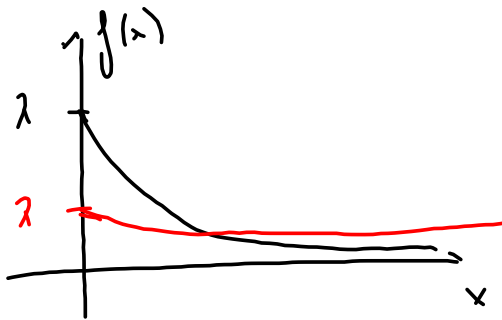
$$= \lim_{c \rightarrow \infty} \left[-\frac{e^{-ac}}{a} + \frac{e^{-a \cdot 0}}{a} \right] = \frac{1}{a}$$

$\downarrow$
 0

Note that if $\lambda > 0$, then $\int_0^{\infty} \lambda e^{-\lambda x} dx = 1$

Definition: A random variable $\bar{X}$ is exponentially distributed with parameter λ if the density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$



Interpretation: $\bar{X}$ typically models waiting time or lifelengths (of a light bulb)

Cumulative distribution: $F(x) = \int_0^x \lambda e^{-\lambda t} dt = \left[-e^{-\lambda t} \right]_0^x = -e^{-\lambda x} + e^0$

$$= \underline{\underline{1 - e^{-\lambda x}}}$$

Survival function: $\bar{F}(x) = P[\bar{X} > x] = 1 - F(x) = 1 - (1 - e^{-\lambda x})$

$$= \underline{\underline{e^{-\lambda x}}}$$

Moment generating function: $M_X(t) = E[e^{tX}] = \int_0^{\infty} \underbrace{e^{tx}}_{a=\lambda-t} \lambda e^{-\lambda x} dx$

$$= \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx = \lambda \frac{1}{\lambda-t} = \frac{\lambda}{\lambda-t} \quad \text{for } t < \lambda.$$

Derivatives: $M'_X(t) = \left(\frac{\lambda}{\lambda-t} \right)' = \lambda \left(-\frac{1}{(\lambda-t)^2} \right) (-1) = \frac{\lambda}{(\lambda-t)^2}$

$$= \lambda (\lambda-t)^{-2}$$

$$M''_X(t) = [\lambda (\lambda-t)^{-2}]' = \lambda (-2) (\lambda-t)^{-3} (-1) = \frac{2\lambda}{(\lambda-t)^3}$$

Expectation: $E[X] = M'_X(0) = \frac{\lambda}{(\lambda-0)^2} = \frac{1}{\lambda}$ ✓

$$E[X^2] = M''_X(0) = \frac{2\lambda}{(\lambda-0)^3} = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \frac{1}{\lambda^2}$$

Summing up: An exponential random variable with parameter λ has mean $\frac{1}{\lambda}$ and variance $\frac{1}{\lambda^2}$.

Example: The waiting time in a queue is exponentially distributed with mean 5 minutes. Then $\lambda = \frac{1}{5}$, and the variance is $\frac{1}{\lambda^2} = \frac{1}{(\frac{1}{5})^2} = \underline{\underline{25}}$

Def: A random variable with values in $[0, \infty)$ is called memoryless if for all $s, t > 0$, we have

$$(*) \quad P[\bar{X} > t+s | \bar{X} > t] = P[\bar{X} > s].$$

Prob that the lifetime is more than $t+s$ given that it is more than t

Prob that the lifetime is more than s .

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Reformulate (*):

$$\frac{P[\bar{X} > t+s \cap \bar{X} > t]}{P[\bar{X} > t]} = P[\bar{X} > s]$$

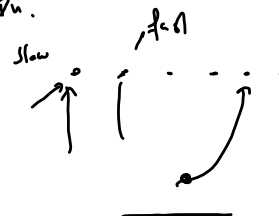
$$\frac{P[\bar{X} > t+s]}{P[\bar{X} > t]} = P[\bar{X} > s] \Rightarrow P[\bar{X} > t+s] = P[\bar{X} > t] P[\bar{X} > s]$$

Prop: Exponential distributions are memoryless:

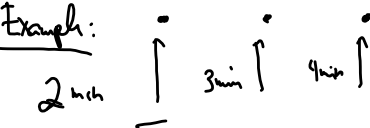
$$\begin{aligned} \text{Proof: } P[\bar{X} > t+s] &= \bar{F}(t+s) = e^{-\lambda(t+s)} = e^{-\lambda t} \cdot e^{-\lambda s} \\ &= \bar{F}(t) \bar{F}(s) = P[\bar{X} > t] P[\bar{X} > s] \end{aligned}$$

Theorem: Assume that $X_1, X_2, \dots, X_n$ are independent and exponentially distributed random variables with parameters $\lambda_1, \lambda_2, \dots, \lambda_n$. Then $\bar{X} = \min\{X_1, X_2, \dots, X_n\}$ is exponentially distributed with parameter $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_n$.

Proof:



Example:



$$\lambda_1 = \frac{1}{2} \quad \lambda_2 = \frac{1}{3} \quad \lambda_3 = \frac{1}{4}$$

$$\bar{X} = \min\{X_1, X_2, X_3\}$$

$$\lambda = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{13}{12}$$

$$\text{Mean waiting: } \frac{1}{\lambda} = \frac{12}{13}$$

Proof: $\bar{X} = \min\{X_1, \dots, X_n\}$

$$\begin{aligned} \bar{F}(x) &= P[\bar{X} > x] = P[X_1 > x \cap X_2 > x \cap \dots \cap X_n > x] = P[X_1 > x] P[X_2 > x] \dots P[X_n > x] \\ &= e^{-\lambda_1 x} e^{-\lambda_2 x} \dots e^{-\lambda_n x} = e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_n)x} \\ &\quad \text{survival function of an exponential r.v. with parameter } \lambda = \lambda_1 + \lambda_2 + \dots + \lambda_n. \end{aligned}$$

Question: $\bar{X}_1 \left\{ \begin{array}{cc} \text{slow} & \text{fast} \end{array} \right\} \bar{X}_2$
 $\lambda_1 \qquad \qquad \qquad \lambda_2$

Theorem: Assume that $\bar{X}_1$ and $\bar{X}_2$ are independent and exponentially distributed with parameters λ_1 and λ_2 . Then

$$P[\bar{X}_1 < \bar{X}_2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Proof:
$$P[\bar{X}_1 < \bar{X}_2] = \int_0^{\infty} \underbrace{P[\bar{X}_2 > x]}_{e^{-\lambda_2 x}} \underbrace{\lambda_1 e^{-\lambda_1 x}}_{\lambda_1 e^{-\lambda_1 x}} dx$$

$$= \int_0^{\infty} e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx = \lambda_1 \int_0^{\infty} e^{-(\lambda_1 + \lambda_2)x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$a = \lambda_1 + \lambda_2$
 $\frac{1}{\lambda_1 + \lambda_2}$

Example: Cheap lightbulb: Expected lifetime 5000 hours } exponentially distributed.
 Expensive " : " : 10000 hours

Lifetime of cheap: $\bar{X}_1$ $\lambda_1 = \frac{1}{5000}$

" expensive: $\bar{X}_2$ $\lambda_2 = \frac{1}{10000}$

$$P[\bar{X}_1 < \bar{X}_2] = \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{\frac{1}{5000} \cdot 10000}{\left(\frac{1}{5000} + \frac{1}{10000}\right) \cdot 10000} = \frac{2}{2+1} = \frac{2}{3}$$

Gamma distribution

Gamma function:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \quad \alpha > 0.$$

If $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$

Def: The gamma (α, λ) -distribution has density

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \leftarrow \alpha=1 \Rightarrow \text{exponential distribution} \\ 0 & \text{for } x \leq 0 \end{cases}$$

Let us check that this really is a distribution function:

$$\int_0^{\infty} f(x) dx = \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

New variable:

$$u = \lambda x \Rightarrow du = \lambda dx$$

$$= \int_0^{\infty} \frac{1}{\Gamma(\alpha)} \underbrace{(\lambda x)^{\alpha-1}}_u e^{-u} du = \frac{1}{\Gamma(\alpha)} \overbrace{\int_0^{\infty} u^{\alpha-1} e^{-u} du}^{\Gamma(\alpha)} = \underline{\underline{1}}$$

Moment generating function: X is gamma (α, λ) -distributed

$$\begin{aligned}
 M_X(t) &= E[e^{tX}] = \int_0^{\infty} e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx \\
 &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(\lambda-t)x} dx = \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \int_0^{\infty} \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx \\
 &= \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \quad \text{for } t < \lambda.
 \end{aligned}$$

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(α, λ-t) gamma density

Hence: $M_X(t) = \frac{\lambda^\alpha}{(\lambda-t)^\alpha} = \lambda^\alpha (\lambda-t)^{-\alpha}$

Derivatives: $M_X'(t) = \lambda^\alpha (-\alpha)(\lambda-t)^{-\alpha-1} (-1) = \alpha \lambda^\alpha (\lambda-t)^{-\alpha-1}$

$$\begin{aligned}
 M_X''(t) &= \alpha \lambda^\alpha (-\alpha-1)(\lambda-t)^{-\alpha-2} (-1) \\
 &= \alpha(\alpha+1) \lambda^\alpha (\lambda-t)^{-\alpha-2} = \frac{\alpha(\alpha+1)\lambda^\alpha}{(\lambda-t)^{\alpha+2}}
 \end{aligned}$$

Expectation $E[X] = M_X'(0) = \alpha \lambda^\alpha (\lambda)^{-\alpha-1} = \frac{\alpha \lambda^\alpha}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}$

$$E[X^2] = M_X''(0) = \frac{\alpha(\alpha+1)\lambda^\alpha}{\lambda^{\alpha+2}} = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$\begin{aligned}
 \text{Var}(X) &= E[X^2] - E[X]^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} \\
 &= \frac{\cancel{\alpha^2}}{\lambda^2} + \frac{\alpha}{\lambda^2} - \frac{\cancel{\alpha^2}}{\lambda^2} = \frac{\alpha}{\lambda^2}
 \end{aligned}$$

Assume that $X_1, X_2, \dots, X_n$ are independent and exponentially distributed with parameter λ . Then the moment generating function of

$X_1 + X_2 + \dots + X_n$ is

$$\begin{aligned} E[e^{t(X_1 + \dots + X_n)}] &= E[e^{tX_1} \cdot e^{tX_2} \dots e^{tX_n}] \stackrel{\text{independence}}{=} \\ &= \underbrace{E[e^{tX_1}]}_{\frac{\lambda}{\lambda-t}} \underbrace{E[e^{tX_2}]}_{\frac{\lambda}{\lambda-t}} \dots \underbrace{E[e^{tX_n}]}_{\frac{\lambda}{\lambda-t}} = \frac{\lambda^n}{(\lambda-t)^n} \quad \text{gamma}(n, \lambda) \end{aligned}$$

$\nearrow$
 $\uparrow$