

Last line:

f_i = the probability that the MC returns to i when started in i

Two cases:

$f_i = 1$: The MC is recurrent and will visit i an infinite no. of times

$$\sum_{n=1}^{\infty} P_{ii}^n = \infty$$

$f_i < 1$: The MC is transient, will only visit i a finite no. of times

$$\sum_{n=1}^{\infty} P_{ii}^n < \infty$$

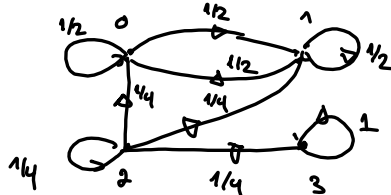
Also: If $i \leftrightarrow j$, then they are either both recurrent or both transient.

Proposition: If S is finite, not all states can be transient.

Proof: Start the chain at 0 and look at how many times it visits each state. As there are only finitely many states, there must be one that is hit infinitely many times with positive probability, say i . This state has to be recurrent, because if you wait till it hits i for the first time, it will come back to i with pos. prob. an infinite times.

Corollary: If a MC is irreducible and has a finite state space, then all states are recurrent.

Proof: Since the state space is finite, there is at least one recurrent state. Since all states communicate, they must all be recurrent.

Example

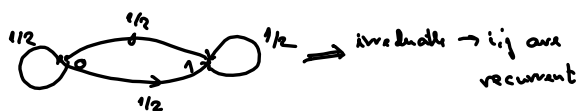
Classes: $\{0, 1\}$, $\{2\}$, $\{3\}$

3: Recurrent.

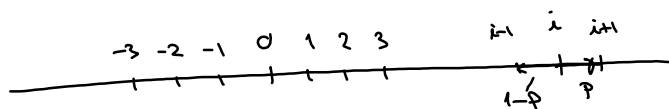
$$\sum_{n=1}^{\infty} P_{22}^n = \left(\frac{1}{4}\right)^n, \quad \sum_{n=1}^{\infty} P_{22}^n = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n < \infty \text{ transient}$$

0 and 1

recurrent



Example: Random walk on the line $S = \mathbb{Z}$



irreducible: Is 0 transient or recurrent?

$$\sum P_{00}^n$$

$$n \text{ is odd} \Rightarrow P_{00}^n = 0$$

$$n \text{ is even: } n = 2k: P_{00}^{2k} = \binom{2k}{k} p^k (1-p)^k$$

Does $\sum_{k=0}^{\infty} \binom{2k}{k} p^k (1-p)^k$ converge?

Ratio test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \begin{cases} < 1 & \text{converge} \\ = 1 & \text{Hell's horse} \\ > 1 & \text{diverge} \end{cases}$

$$\lim_{k \rightarrow \infty} \frac{\binom{2k+2}{k+1} p^{k+1} (1-p)^{k+1}}{\binom{2k}{k} p^k (1-p)^k} = \lim_{k \rightarrow \infty} \frac{(2k+2)(2k+1) \dots (k+1)}{k!} \frac{p(1-p)}{1 \cdot 2 \dots k} = \lim_{k \rightarrow \infty} \frac{(2k+2)(2k+1)}{(k+1)(k+1)} p(1-p)$$

$$= 4p(1-p)$$

We have that the limit is

$$\begin{cases} 1 & \text{for } p = 1/2 \text{ (don't know)} \\ < 1 & \text{for } p \neq 1/2, \text{ transience} \end{cases}$$


Heavy machinery: Stirling's formula: $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$
 $(a_n \sim b_n \text{ means } \frac{a_n}{b_n} \rightarrow 1 \text{ as } n \rightarrow \infty)$

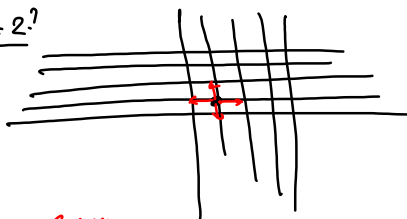
Closer look at $p = 1/2$:

$$P_{00}^{2k} = \binom{2k}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^k = \frac{(2k)!}{k! k!} \left(\frac{1}{2}\right)^{2k} \sim \frac{\sqrt{2\pi 2k} \left(\frac{2k}{e}\right)^{2k}}{\sqrt{2\pi k} \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \left(\frac{k}{e}\right)^k} = \frac{1}{\sqrt{\pi k}}$$

$$\text{The series } \sum \frac{1}{\sqrt{\pi k}} = \frac{1}{\sqrt{\pi}} \sum \frac{1}{k^{1/2}} \text{ diverges } \sum \frac{1}{k^p} \quad p \leq 1$$

By the limit comparison test $\sum P_{00}^{2k}$ also diverges, hence for $p = 1/2$ the MC is recurrent.

What happens if $d = 2$?



$$\sum P_{00}^n \sim \sum \frac{1}{n}$$

$$3D \sum P_{00}^n \sim \sum \frac{1}{n^{3/2}} \text{ conv.}$$