

Failure functions

X positive random variable $\begin{cases} f & \text{density} \\ F & \text{cumulative dist. func.} \end{cases} \quad F' = f.$

Failure function:

$$r(t) = \frac{f(t)}{1 - F(t)}$$

Think of X as lifetime: Given that the component has lasted till t , what is the prob. that it will break down in the interval $(t, t + \Delta t)$

$$\begin{aligned} P[X \in (t, t + \Delta t) \mid X > t] &= \frac{P[X \in (t, t + \Delta t) \cap \cancel{X \leq t}]}{P[X > t]} \\ &= \frac{P[X \in (t, t + \Delta t)]}{P[X > t]} \approx \frac{f(t) \Delta t}{1 - F(t)} = r(t) \Delta t \end{aligned}$$

Exponential distribution: $f(t) = \lambda e^{-\lambda t}$ $F(t) = 1 - e^{-\lambda t}$

$$r(t) = \frac{\lambda e^{-\lambda t}}{\underbrace{1 - (1 - e^{-\lambda t})}_{e^{-\lambda t}}} = \lambda$$

Reconstructing the distribution from the failure function

Given r , find F .

Note $r(s) = \frac{f(s)}{1-F(s)}$ ↖ $F'(s)$

Integrate: $\int_0^t r(s) ds = \int_0^t \frac{f(s)}{1-F(s)} ds = \left[-\ln(1-F(s)) \right]_0^t$

$$= -\ln(1-F(t)) + \ln(1-F(0)) \stackrel{0}{=} = -\ln(1-F(t))$$

Equation: $\int_0^t r(s) ds = -\ln(1-F(t))$

$$e^{-\int_0^t r(s) ds} = e^{\ln(1-F(t))} = 1-F(t)$$

$$F(t) = 1 - e^{-\int_0^t r(s) ds}$$

Exponential dist: $1 - e^{-\int_0^t \lambda ds} = 1 - e^{-\lambda t} = F(t)$.

Sums of exponential r.v.

If $X_1, X_2, \dots, X_n$ are independent, exponential r.v. with parameters $\lambda_1, \lambda_2, \dots, \lambda_n$

what is $X_1 + X_2 + \dots + X_n$?

Solved when $\lambda_1 = \lambda_2 = \dots = \lambda_n$ ($\Gamma(n, \lambda)$):

$n=2$: X and Y exponential, independent

$\downarrow \quad \downarrow$
 $\lambda \quad \mu$

$$\begin{aligned} f_{X+Y}(t) &= \int_{-\infty}^{\infty} \underbrace{f_X(s)}_{\lambda e^{-\lambda s}} \underbrace{f_Y(t-s)}_{\mu e^{-\mu(t-s)}} ds = \int_0^t \lambda e^{-\lambda s} \mu e^{-\mu(t-s)} ds \\ &= \lambda \mu e^{-\mu t} \int_0^t e^{(\mu-\lambda)s} ds = \lambda \mu e^{-\mu t} \left[\frac{e^{(\mu-\lambda)s}}{(\mu-\lambda)} \right]_0^t \\ &= \frac{\lambda \mu e^{-\mu t}}{\mu-\lambda} \left[e^{\mu t - \lambda t} - 1 \right] = \frac{\lambda \mu}{\mu-\lambda} e^{-\lambda t} + \frac{\lambda \mu}{\lambda-\mu} e^{-\mu t} \\ &= \frac{\mu}{\mu-\lambda} \lambda e^{-\lambda t} + \frac{\lambda}{\lambda-\mu} \mu e^{-\mu t} \end{aligned}$$

$$\begin{aligned} \underline{n=3}: f_{X_1+X_2+X_3}(t) &= f_{(X_1+X_2)+X_3}(t) = \int_0^t \underbrace{f_{X_1+X_2}(s)}_{\lambda_2 e^{-\lambda_2 s}} \underbrace{f_{X_3}(t-s)}_{\lambda_3 e^{-\lambda_3(t-s)}} ds \\ &\quad \downarrow \quad \downarrow \quad \downarrow \\ &\quad \lambda_1 \quad \lambda_2 \quad \lambda_3 \end{aligned}$$

$$\begin{aligned} &= \int_0^t \left(\frac{\lambda_2}{\lambda_2-\lambda_1} \lambda_1 e^{-\lambda_1 s} + \frac{\lambda_1}{\lambda_1-\lambda_2} \lambda_2 e^{-\lambda_2 s} \right) \lambda_3 e^{-\lambda_3(t-s)} ds \\ &= \frac{\lambda_2}{\lambda_2-\lambda_1} \int_0^t \lambda_1 e^{-\lambda_1 s} \lambda_3 e^{-\lambda_3(t-s)} ds + \frac{\lambda_1}{\lambda_1-\lambda_2} \int_0^t \lambda_2 e^{-\lambda_2 s} \lambda_3 e^{-\lambda_3(t-s)} ds \\ &= \frac{\lambda_2}{\lambda_2-\lambda_1} \left[\frac{\lambda_3}{\lambda_3-\lambda_1} \lambda_1 e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1-\lambda_3} \lambda_3 e^{-\lambda_3 t} \right] + \dots \\ &= \frac{\lambda_2 \lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} \lambda_1 e^{-\lambda_1 t} + \frac{\lambda_1 \lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} \lambda_2 e^{-\lambda_2 t} + \frac{\lambda_1 \lambda_2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} \lambda_3 e^{-\lambda_3 t} \end{aligned}$$

General case:

$$f_{X_1+X_2+\dots+X_n}(t) = \sum_{i=1}^n \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \lambda_i e^{-\lambda_i t}$$