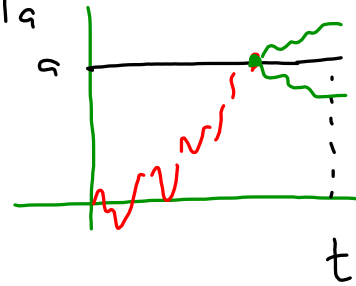
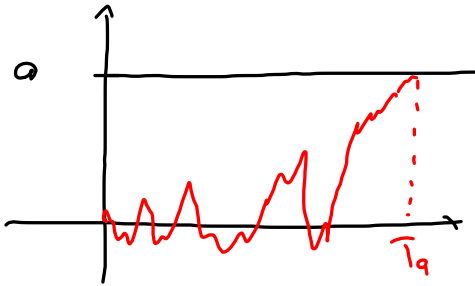


Calculations with Brownian motion

~~X~~ - Brownian motion, $a > 0$

$T_a = \min\{s: \bar{X}(s) \geq a\}$ Want to find the distribution of T_a



$$P[\bar{X}_t \geq a] = \underbrace{P[\bar{X}_t \geq a | T_a \leq t]}_{1/2} P[T_a \leq t] + \underbrace{P[\bar{X}_t \geq a | T_a > t]}_0 P[T_a > t]$$

$$= \frac{1}{2} P[T_a \leq t]$$

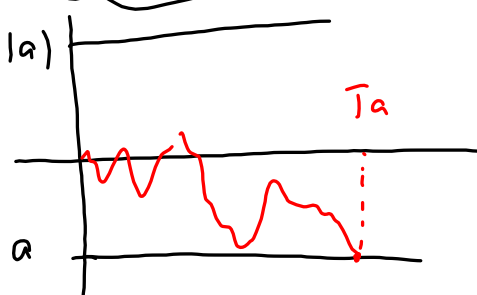
Hence $P[T_a \leq t] = 2 P[\bar{X}_t \geq a] = 2 \cdot \int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy$

gaussian mean 0 variance t

$y = \frac{x}{\sqrt{t}}, \quad dy = \frac{dx}{\sqrt{t}}$

$-\frac{y^2}{2} = -\frac{x^2}{2t}$

$$= 2 \int_{a/\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \rightarrow 1 \text{ as } t \rightarrow \infty.$$



By symmetry

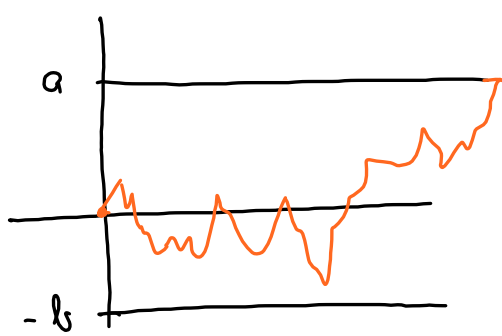
$$T_a \stackrel{d}{=} T_{|a|}$$

$$P[T_a \leq t] = P[T_{|a|} \leq t]$$

$$= 2 \int_{|a|/\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

Maximum: $\max_{s \leq t} \bar{X}_s, \quad a > 0$

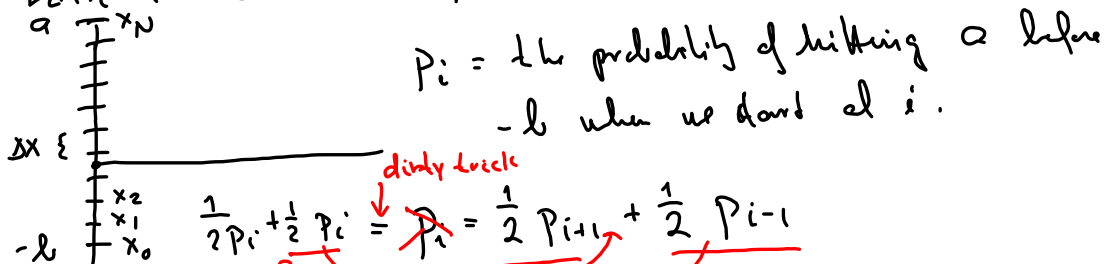
$$P[\max_{s \leq t} \bar{X}_s \leq a] = P[T_a \leq t] = 2 \int_{a/\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$



a, b

What is the probability that X hits a before $-b$?

Don't have the machinery, do the discrete case



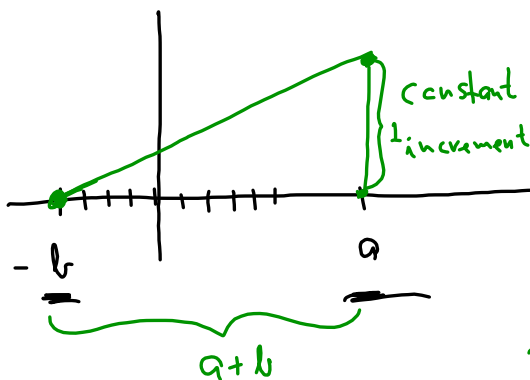
P_i = the probability of hitting a before $-b$ when we start at i .

$$\frac{1}{2} P_i + \frac{1}{2} P_i = \cancel{P_i} = \frac{1}{2} P_{i+1} + \frac{1}{2} P_{i-1}$$

dinky little

$$\frac{1}{2} P_i - \frac{1}{2} P_{i-1} = \frac{1}{2} P_{i+1} - \frac{1}{2} P_i$$

$$\underline{P_i - P_{i-1} = P_{i+1} - P_i = \Delta P}$$



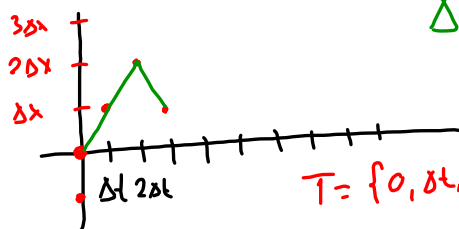
$$P_0 = 0$$

$$P_N = 1$$

$$p(x) = \frac{1}{a+b} (x+b)$$

$$p(0) = \frac{1}{a+b} (0+b) = \underline{\underline{\frac{b}{a+b}}}$$

Random walk

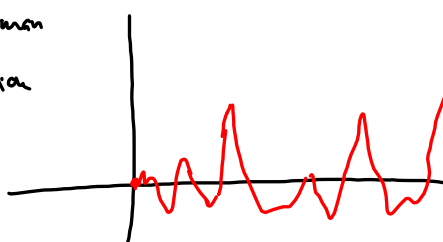


$$\Delta x = \sigma \sqrt{\Delta t}$$

$$\bar{X}: \Omega \times T \rightarrow S$$

$$T = \{0, \Delta t, 2\Delta t, \dots\} \quad S = \{ \dots, -\Delta x, 0, \Delta x, 2\Delta x, \dots \}$$

Brownian motion



$$B: \Omega \times [0, \infty) \rightarrow \mathbb{R}$$

$$\bar{X} \Rightarrow B \text{ as } \Delta t \rightarrow 0$$

↓
weak convergence.

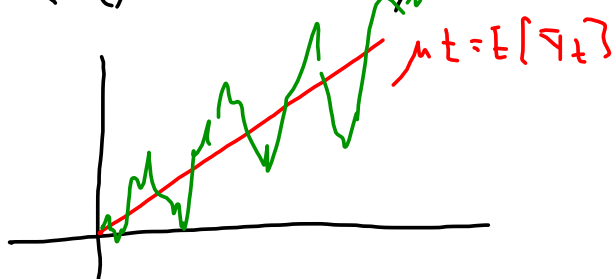
Some variants of Brownian motion

B_t be standard Brownian motion, μ, σ are constants.

$$\underline{Y_t = \mu t + \sigma B_t} \quad \left(\begin{array}{l} \text{a diffusion with drift } \mu \text{ and} \\ \text{variance coefficient } \sigma \end{array} \right)$$

$$E[Y_t] = E[\mu t + \sigma B_t] = \mu t$$

$$\text{Var}(Y_t) = E[(Y_t - \mu t)^2] = E[(\sigma B_t)^2] = \sigma^2 t$$



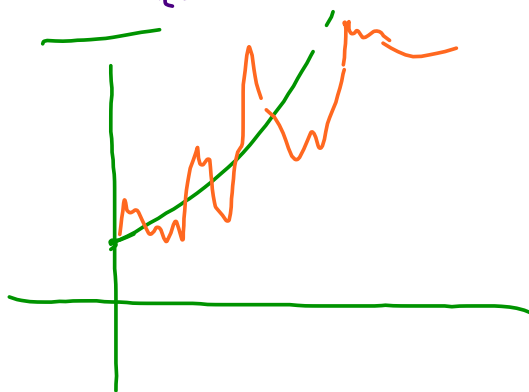
Geometric Brownian motion:

$$\underline{X_t = e^{Y_t} = e^{\mu t + \sigma B_t}} > 0 \quad \text{normal.}$$

Distribution of X_t : lognormal since $\log X_t = \mu t + \sigma B_t$

Mean of X_t : normal with mean μt and variance $\sigma^2 t$.

$$\underline{E[X_t] = E[e^{Y_t}] = M(1) = e^{\mu t + \frac{\sigma^2}{2} t} = e^{(\mu + \frac{\sigma^2}{2}) t}}$$



A discrete approximation

ΔB_t random walk Δt , $\underline{\Delta x^2 = \Delta t}$ ($\Delta t \rightarrow 0$)

$$\bar{X}_0 = X_0 \in \mathbb{R}$$

$$\bar{X}_{\Delta t} = X_0 (1 + a\Delta t + b\Delta B(0)) = \begin{cases} X_0 (1 + a\Delta t + b\sqrt{\Delta t}) & \text{prob } 1/2 \\ X_0 (1 + a\Delta t - b\sqrt{\Delta t}) & \end{cases}$$

$$\bar{X}_{2\Delta t} = \bar{X}_{\Delta t} (1 + a\Delta t + b\Delta B(\Delta t))$$

$$= X_0 (1 + a\Delta t + b\Delta B(0)) (1 + a\Delta t + b\Delta B(\Delta t))$$

$\vdots$

$$\bar{X}_t = X_0 \prod_{s=0}^{t-\Delta t} (1 + a\Delta t + b\Delta B(s))$$

What does this have to do with GBM

$$\begin{aligned} \bar{X}_t &= X_0 e^{\sum_{s=0}^{t-\Delta t} \ln(1 + a\Delta t + b\Delta B(s))} \\ &= X_0 e^{\sum_{s=0}^{t-\Delta t} \ln(1 + \underbrace{a\Delta t + b\Delta B(s)}_{\text{small}})} \quad \ln(1+x) \sim x - \frac{x^2}{2} \\ &\approx X_0 e^{\sum_{s=0}^{t-\Delta t} \left[\underbrace{a\Delta t + b\Delta B(s)}_{\text{small}} - \frac{1}{2} (a\Delta t + b\Delta B(s))^2 \right]} \\ &= X_0 e^{at + bB(t) - \frac{1}{2} \left[\sum_{s=0}^{t-\Delta t} a^2 \Delta t^2 + 2ab \Delta t \Delta B(s) + \underbrace{b^2 \Delta B(s)^2}_{\Delta x^2 = \Delta t} \right]} \\ &\approx X_0 e^{at + bB(t) - \frac{1}{2} b^2 t} \end{aligned}$$

geometric BM.

$$\begin{aligned} 2ab \Delta t \sum_{s=0}^{t-\Delta t} \Delta B(s) &= 2ab \Delta t B(t) \quad \downarrow 0 \\ b^2 \sum_{s=0}^{t-\Delta t} \Delta t &= b^2 t \end{aligned}$$