

Limiting probabilities

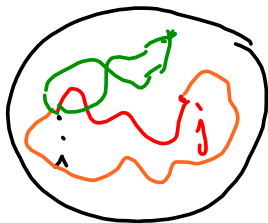
Last time:

f_i = the prob of returning to i if we start in i

f_{ij} = the prob of hitting j if we start in i
 $= P[\text{that there is an } n \text{ s.t. } X_n = j \mid X_0 = i]$

Proposition: If $i \leftrightarrow j$ and i is recurrent, then $f_{ij} = 1$.

Proof: Since $i \leftrightarrow j$, there is a n s.t. $\tilde{P}_{ij}^n > 0$.



We have a success at the first attempt if $X_n = j$ (happens with prob. $\tilde{P}_{ij}^n$).

Let the chain return to i .

We have a success at the second attempt if X is at j n steps after the first return

Prob: $(1 - \tilde{P}_{ij}^n) \tilde{P}_{ij}^n$ - if first success now.

Let the chain return to i .

Success of the third attempt if X is at j n steps later and no previous success.

Prob $(1 - \tilde{P}_{ij}^n)^2 \tilde{P}_{ij}^n$

$\vdots$
 Success at the k -th attempt (and no previous success) is

Prob: $(1 - \tilde{P}_{ij}^n)^{k-1} \tilde{P}_{ij}^n$

Total prob of succeeding at some attempt:

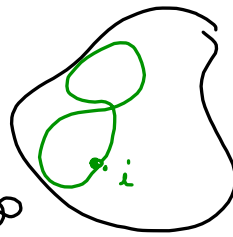
$$\tilde{P}_{ij}^n + (1 - \tilde{P}_{ij}^n) \tilde{P}_{ij}^n + \dots + (1 - \tilde{P}_{ij}^n)^{k-1} \tilde{P}_{ij}^n + \dots = \frac{\tilde{P}_{ij}^n}{1 - (1 - \tilde{P}_{ij}^n)} = \underline{\underline{1}}$$

$\bar{X}$ is recurrent:

$$N_i = \min \{n > 0 : X_n = i\}$$

$$m_i = E[N_i | \bar{X}_0 = i] \text{ (expected first return)}$$

Def: $\bar{X}$ is positive recurrent if $m_i < \infty$
and it is null recurrent if $m_i = \infty$



Expected occupation:

$$\pi_i = \lim_{n \rightarrow \infty} \frac{E[\#\{k \leq n : X_k = i\} | \bar{X}(0) = i]}{n}$$

the proportion of time spent in A at i .

Proposition If $\bar{X}$ is irreducible and recurrent, then

$$\pi_j = \frac{1}{m_j} \quad (\text{when } \frac{1}{\infty} \text{ is interpreted as } 0).$$

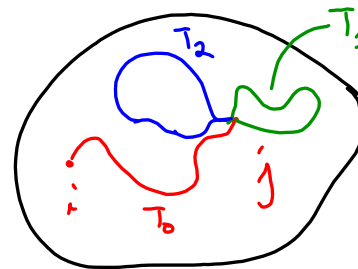
"Plausible argument": Assume we start in state i .

T_0 = number of steps to hit j .

T_1 = number of steps between first and second hit of j .

T_2 = number of steps from second to third hit of j

$\vdots$



$T_1, T_2, T_3, \dots$ are independent, identically distributed, with mean m_j

$$\text{LLN} \quad \frac{T_1 + T_2 + \dots + T_n}{n} \rightarrow m_j$$

Let us look at the situation after the n -th return

$$\frac{n+1}{T_0 + T_1 + T_2 + \dots + T_n} = \frac{1}{\frac{T_0}{n+1} + \frac{1}{n+1} [T_1 + T_2 + \dots + T_n]}$$

time spent in j total time

$$= \frac{1}{\frac{T_0}{n+1} + \frac{1}{n} [T_1 + \dots + T_n] \cdot \frac{n+1}{n}} \rightarrow \frac{1}{m_j}$$

$\downarrow$ 0 $\downarrow$ m_j $\downarrow$ 1 "Hence" $\pi_j = \frac{1}{m_j}$

Prop: If i is positive recurrent and $i \leftrightarrow j$, then j is also positive recurrent.

"Heuristic argument": Since $i \leftrightarrow j$, there is an n such that $P_{ij}^n > 0$. Note that

$\pi_i P_{ij}^n$ = the proportion of time the MC is in i and then n steps later it is in j .

= the proportion of time the MC is in j and n steps before it has been in i .

$\leq$ the proportion of time the MC spends in j = π_j

Hence $\pi_j \geq \pi_i P_{ij}^n > 0$, which shows that j is pos. recurrent.

Theorem: Assume that X is irreducible. If X is positive recurrent, then $\pi_1, \pi_2, \pi_3, \dots$ are the only solutions to the equation

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{for each } j$$

with $\sum_j \pi_j = 1$

Matrix form

$$\vec{\pi} = (\pi_0, \pi_1, \pi_2, \dots)$$

$$\vec{\pi} = \vec{\pi} P \Rightarrow \vec{\pi} (P - I) = 0$$

$\vec{\pi}$ is an invariant distribution.

Explanation of formula:

$\pi_i P_{ij}$ = the proportion of time spent at j and coming from i .

$\sum_i \pi_i P_{ij}$ = the proportion of time spent at j coming from one of the other sites

= the proportion of time spent at j = π_j

Example: R - it rains $\begin{matrix} R & \overline{R} \\ \overline{R} & R \end{matrix} \begin{matrix} 0.7 \\ 0.3 \end{matrix}$
 $\overline{R}$ - it doesn't rain $\begin{matrix} 0.4 & R \\ 0.6 & \overline{R} \end{matrix}$

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \quad \vec{\pi} = (\pi_0, \pi_1)$$

We need $\vec{\pi} = \vec{\pi} P$

$$(\pi_0, \pi_1) \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = (0.7\pi_0 + 0.4\pi_1, 0.3\pi_0 + 0.6\pi_1)$$

Equation: $\left. \begin{matrix} \pi_0 = 0.7\pi_0 + 0.4\pi_1 \\ \pi_1 = 0.3\pi_0 + 0.6\pi_1 \end{matrix} \right\} \pi_0 + \pi_1 = 1$

$$\left. \begin{matrix} 0.3\pi_0 = 0.4\pi_1 \\ 0.3\pi_0 = 0.4\pi_1 \end{matrix} \right\} \Rightarrow \pi_0 = \frac{4}{3}\pi_1$$

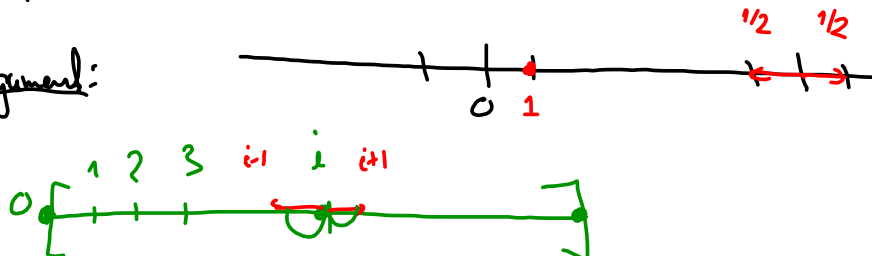
Hence $1 = \pi_0 + \pi_1 = \frac{4}{3}\pi_1 + \pi_1 = \frac{7}{3}\pi_1 \Rightarrow \pi_1 = \frac{3}{7}$ $\rightarrow$ it rains $\frac{3}{7}$ of the time.

$\pi_0 = \frac{4}{7} \rightarrow$ it's going to rain $\frac{4}{7}$ of the time

Corrally: If we have a irreducible MC with a finite state space, all states are positively recurrent.

Example: The symmetric random walk on the integers is null recurrent.

Argument:



t_i = the expected number of steps we need to reach either 0 or n starting at i .

$$t_i = \frac{1}{2}t_{i+1} + \frac{1}{2}t_{i-1} + \underline{1} \quad | \quad 2$$

$$\boxed{t_{i+1} - 2t_i + t_{i-1} = -2} \quad \xrightarrow{\text{MAT-INF 1100}}$$

$$t_0 = 0, t_n = 0 \quad \underline{t_i = n_i - i^2}$$

Hence $t_1 = \underline{n-1}$ The expectation for the full walk must be ∞ , null-recurrent.

