

Markov chains, review II

Last time: $\bar{X}$, recurrent, $N_i = \min\{t > 0, \bar{X}(t) = i\}$

$$m_i = E_i[N_i] = E[N_i | \bar{X}(0) = i]$$

$m_i < \infty$, $\bar{X}$ is pos. recurrent
 $m_i = \infty$, $\bar{X}$ is null recurrent

} class properties

Th: S finite, $\bar{X}$ irreducible, then $\bar{X}$ is pos. recurrent.

Long time occupancy

$$\pi_i = \lim_{n \rightarrow \infty} \frac{\#\{k < n : \bar{X}(k) = i\}}{n} \quad (\text{assuming existence})$$

Theorem: Assume that $\bar{X}$ is irreducible and recurrent. Then

$$\pi_i = \frac{1}{m_i}$$

Theorem: Assume that $\bar{X}$ is irreducible and positive recurrent.

Then $\vec{\pi} = (\pi_0, \pi_1, \dots)$ is the unique solution to

$$\vec{\pi} P = \vec{\pi}$$

such that $\pi_0, \pi_1, \dots > 0$ and $\sum_{i \in S} \pi_i = 1$.

Consequence: $\vec{\pi}$ is a stationary distribution:

If we start $\bar{X}$ with distribution $\vec{\pi}$, the distribution will be π for all times:

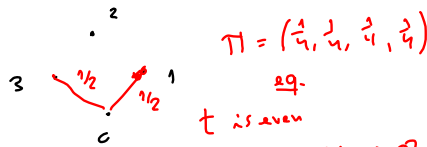
$$P[\bar{X}(t) = i] = \pi_i$$

equilibrium

Question: Will the distribution of $\bar{X}$ (from any starting position) converge to π ; i.e.

$$\lim_{n \rightarrow \infty} P_{ij}^{(n)} = \pi_j ?$$

In general NO:



$$\pi = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

eg.

t is even

$$P[\bar{X}(t) = 0] = P[\bar{X}(t) = 2] = \frac{1}{2}$$

t is odd

$$P[\bar{X}(t) = 1] = P[\bar{X}(t) = 3] = \frac{1}{2}$$

Def: i is periodic if there is a number $d > 1$ such that if $P_{ii}^{(n)} > 0$, then n is divisible by d .

If not, i is aperiodic (class property).

ergodic

Theorem If $\bar{X}$ is irreducible, positive recurrent and aperiodic, then for all i, j

$$\lim_{n \rightarrow \infty} P_{ij}^{(n)} = \pi_j \quad (\text{convergence to equilibrium})$$

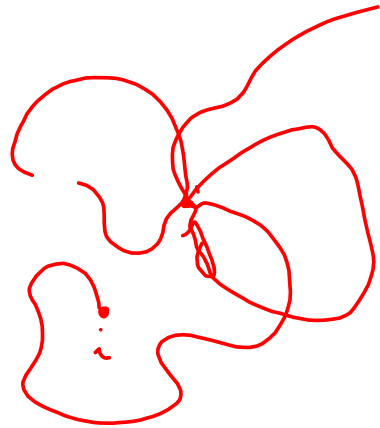
Mean time in transient states

S finite, $\mathbb{X}$ Markov chain

Transient states: $1, 2, 3, \dots, T$

Reduced transition matrix:

$$P_T = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1T} \\ p_{21} & p_{22} & \dots & p_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ p_{T1} & p_{T2} & \dots & p_{TT} \end{pmatrix}$$



$$S = (I - P_T)^{-1}$$

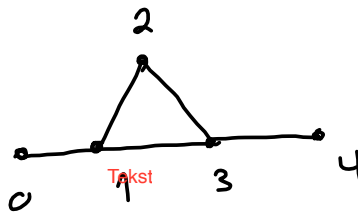
Theorem: S_{ij} = the expected number of visits to j if you start in i

f_{ij} = prob. that starting in i , $\mathbb{X}$ will hit j

$$f_{ij} = \frac{S_{ij} - S_{ji}}{S_{ii}}$$

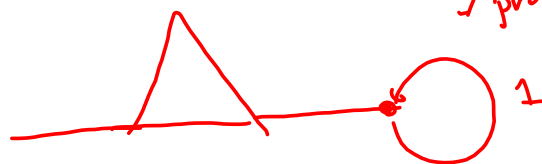
NB Misprint: The denominator should be $s_{\{jj\}}$ and not $s_{\{ii\}}$

Trick: (Assignment)



What is the prob. starting at 0 to hit 2 before you 4.

Change the chain by making 4 sticky



↑
/ prob. of hitting?

X ergodic, stationary distribution $\vec{\pi} = (\pi_0, \pi_1, \dots)$.

$$\overline{AP} = \overline{PQ}$$

Theorem: If there is a vector $\vec{\pi} = (\pi_0, \pi_1, \dots)$ such that $\pi_i > 0$ and $\sum \pi_i = 1$ and for all i, j

$$\pi_i p_{ij} = \pi_j p_{ji}$$

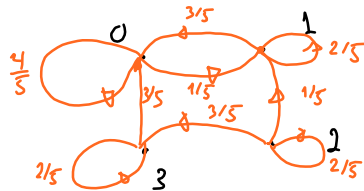
then π is variable with equilibrium state π .

Exam 2022 Problem 1

$$P = \begin{bmatrix} 2q & p & 0 & 0 \\ 3p & q & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 0 & 1/5 \end{bmatrix}$$

a) P transition matrix if $\begin{cases} 2q + p = 1 \\ 3p + q = 1 \end{cases} \Rightarrow q = \frac{2}{5}, p = \frac{1}{5}$

b) Draw the state diagram



$$P = \begin{bmatrix} 4/5 & 1/5 & 0 & 0 \\ 3/5 & 2/5 & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 0 & 1/5 \end{bmatrix}$$

P_{30}

c) $P[\bar{X}_3 = 0, \bar{X}_2 = 3 | \bar{X}_0 = 2] = P_2[\bar{X}_3 = 0 \wedge \bar{X}_2 = 3]$

We get

$$P_2[\bar{X}_3 = 0 \wedge \bar{X}_2 = 3] = P_2[\bar{X}_3 = 0 | \bar{X}_2 = 3] P_2[\bar{X}_2 = 3] = \frac{8}{25} \cdot \frac{3}{5} = \frac{24}{125}$$

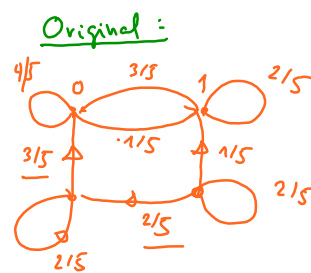
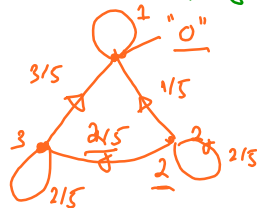
Two step transition matrix (should have done this one first!)

$$P^{(2)} = P \cdot P = \begin{bmatrix} 4/5 & 1/5 & 0 & 0 \\ 3/5 & 2/5 & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 0 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} \text{same} \\ \text{shift} \end{bmatrix} = \begin{bmatrix} 11/25 & 6/25 & 0 & 0 \\ 18/25 & 7/25 & 0 & 0 \\ 9/25 & 4/25 & 4/5 & 1/5 \\ 18/25 & 3/25 & 0 & 1/5 \end{bmatrix}$$

d) $P[\bar{X}_k = \{0, 1\} \text{ for some } 1 \leq k \leq 4 | \bar{X}_0 = 2]$

$$= P_2[\bar{X}_k = \{0, 1\} \text{ for some } k \leq 4]$$

Make a new stepping at $\{0, 1\}$



$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 2/5 & 2/5 \\ 3/5 & 0 & 2/5 \end{pmatrix}$$

$$P_2[\bar{X}_k = \{0, 1\} \text{ for some } k \leq 4]$$

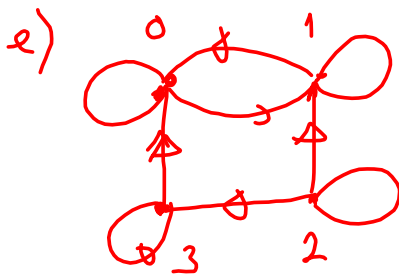
$$= Q_2[\bar{X}_4 = \text{"0"}] = Q_{2 \text{"0"}}^4$$

Need Q^4 :

$$Q^2 = \begin{bmatrix} 1 & 0 & 0 \\ 13/25 & 4/5 & 8/25 \\ 21/25 & 0 & 4/25 \end{bmatrix}$$

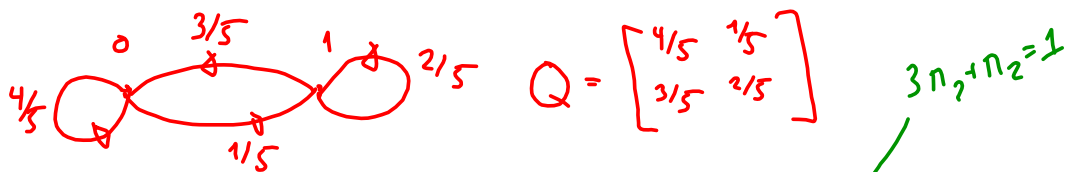
$$Q^4 = Q^2 \cdot Q^2$$

$$= \begin{bmatrix} \text{"0"} \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0.672 \end{bmatrix}$$



Classes: $\{0,1\}$ - recurrent
 $\{2\}$ - transient
 $\{3\}$ - transient

f) Find the stationary distribution given that you have entered $\{0,1\}$: $\vec{\pi} = (\pi_1, \pi_2)$ $\vec{\pi} Q = \vec{\pi}$



$$Q = \begin{bmatrix} 4/5 & 1/5 \\ 3/5 & 2/5 \end{bmatrix}$$

$$3\pi_1 + \pi_2 = 1$$

$$(\pi_1, \pi_2) \begin{pmatrix} 4/5 & 1/5 \\ 3/5 & 2/5 \end{pmatrix} = (\pi_1, \pi_2)$$

$$\frac{\pi_1 + \pi_2 = 1}{\hookrightarrow \left(\frac{3}{4}, \frac{1}{4} \right)}$$

$$\left. \begin{array}{l} 5 \mid \frac{4}{5}\pi_1 + \frac{3}{5}\pi_2 = \pi_1 \\ 5 \mid \frac{1}{5}\pi_1 + \frac{2}{5}\pi_2 = \pi_2 \end{array} \right\} \Rightarrow \left. \begin{array}{l} 3\pi_1 = \pi_2 \\ \pi_1 = 3\pi_2 \end{array} \right\} \pi_1 = 3\pi_2$$

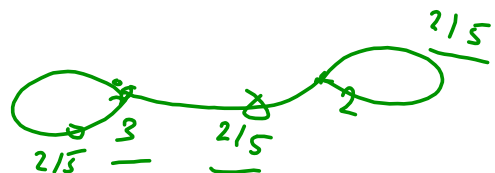
g) s_{23} = expected number of visits to 3 starting in 2.

$$S = (I - P_T)^{-1}, \quad \underline{s_{23}} =$$

$$P_T = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ 0 & \frac{2}{5} \end{bmatrix}$$

$$I - P_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ 0 & \frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ 0 & \frac{3}{5} \end{bmatrix} \Rightarrow S = (I - P_T)^{-1} =$$



$$= \begin{bmatrix} \frac{5}{3} & \frac{10}{9} \\ 0 & \frac{5}{3} \end{bmatrix} \quad s_{23} = \underline{\underline{\frac{10}{9}}}$$