

MC in continuous time

Two diagrams:

(i) Semigroup: $P^t(i, j) = P[\sum_{k=0}^t X(k) = j | X(0) = i]$

(ii) Spectral: v_i - value, time prob P_{ij}



Independent values:

$q_{ij} = v_i P_{ij}$ - the value of leaving i for j .

$$\sum_j q_{ij} = \sum_j v_i P_{ij} = v_i$$

Recall: (i) $\lim_{h \rightarrow 0} \frac{1 - P^h(i, i)}{h} = v_i$

(ii) $\lim_{h \rightarrow 0} \frac{P^h(i, j)}{h} = q_{ij}$ when $i \neq j$

(iii) $P^{t+s}(i, j) = \sum_k P^t(i, k) P^s(k, j)$

Kolmogorov's equations:

(i) The backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

(ii) The forward equations (under suitable conditions, a.s. if from any state you can just go immediately to a finite set of other states):

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

Proof: (i) $P'_{ij}(t) = \lim_{h \rightarrow 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h}$

$$= \lim_{h \rightarrow 0} \frac{\sum_{k \neq i} P_{ik}(t) P_{kj}(t+h) - P_{ij}(t)}{h}$$

$$= \lim_{h \rightarrow 0} \left[\sum_{k \neq i} \frac{P_{ik}(t)}{h} (P_{kj}(t+h) - P_{kj}(t)) + \frac{P_{ij}(t) - P_{ij}(t)}{h} \right]$$

$$= \sum_{k \neq i} q_{ik} P_{kj}(t) - P_{ij}(t) \lim_{h \rightarrow 0} \frac{1 - P_{ii}(h)}{h}$$

$$= \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

(ii) $P'_{ij}(t) = \lim_{h \rightarrow 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h}$

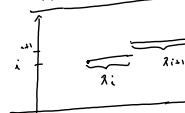
$$= \lim_{h \rightarrow 0} \frac{\sum_{k \neq j} P_{ik}(t) P_{kj}(t+h) - P_{ij}(t)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sum_{k \neq j} P_{ik}(t) P_{kj}(t) + P_{ij}(t) P_{jj}(t+h) - P_{ij}(t)}{h}$$

$$= \sum_{k \neq j} P_{ik}(t) \lim_{h \rightarrow 0} \frac{P_{kj}(t+h) - P_{kj}(t)}{h} + P_{ij}(t) \lim_{h \rightarrow 0} \frac{P_{jj}(t+h) - P_{jj}(t)}{h}$$

$$= \sum_{k \neq j} P_{ik}(t) q_{kj} - v_j P_{ij}(t)$$

Application to pure birth process:



What to compute:

$P_{ij}(t)$ - the prob that if you start with i individuals, then at time t , you will have j individuals

Immediate: If $j < i$, $P_{ij}(t) = 0$.

$$\text{If } j = i, P_{ii}(t) = e^{-\lambda_i t}$$

We are looking for $P_{ij}(t)$ when $j > i$

We start with Kolmogorov's forward equations:

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

In this case (for $j > i$):

$$P'_{ij}(t) = v_{j-1} P_{i, j-1}(t) - v_j P_{ij}(t)$$

$$y'(t) = \lambda_{j-1} y(t) - \lambda_j y(t)$$

$$y'(t) + \lambda_j y(t) = \lambda_{j-1} P_{i, j-1}(t)$$

$$y'(t) + \lambda_j y(t) = \lambda_{j-1} P_{i, j-1}(t)$$

$$e^{\lambda_j t} y'(t) + \lambda_j e^{\lambda_j t} y(t) = \lambda_{j-1} e^{\lambda_j t} P_{i, j-1}(t)$$

$$(e^{\lambda_j t} y(t))' = \lambda_{j-1} e^{\lambda_j t} P_{i, j-1}(t)$$

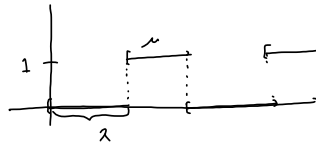
Integrate: $\int_0^t (e^{\lambda_j s} y(s))' ds = \int_0^t \lambda_{j-1} e^{\lambda_j s} P_{i, j-1}(s) ds$

$$\left[e^{\lambda_j t} y(t) \right]_0^t = \lambda_{j-1} \int_0^t e^{\lambda_j s} P_{i, j-1}(s) ds$$

$$e^{\lambda_j t} y(t) = \lambda_{j-1} \int_0^t e^{\lambda_j s} P_{i, j-1}(s) ds \Big|_{t=0}^{t=t}$$

In general, if the state space has n elements, the Kolmogorov equations will give you a system of n linked diff. eq.

Application: A machine has two states, working and not working



$$v_0 = \lambda, v_1 = \mu, p_{01} = 1, p_{10} = 1$$

$$q_{01} = v_0 p_{01} = \lambda, q_{10} = v_1 p_{10} = \mu$$

What is the prob that the machine is working at time t
 $P_{00}(t)$ $j=0$

Backward equation: $P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$

In this case

$$P'_{00}(t) = q_{01} P_{10}(t) - v_0 P_{00}(t)$$

$$= \lambda P_{10}(t) - \lambda P_{00}(t)$$

$$P'_{10}(t) = q_{10} P_{00}(t) - v_1 P_{10}(t)$$

$$= \mu P_{00}(t) - \mu P_{10}(t)$$

Systematic:

$$\begin{cases} P'_{00}(t) = -\lambda P_{00}(t) + \lambda P_{10}(t) \\ P'_{10}(t) = \mu P_{00}(t) - \mu P_{10}(t) \end{cases}$$

Now write

$$\begin{cases} x(t) = P_{00}(t) \\ y(t) = P_{10}(t) \end{cases}$$

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\vec{r}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

$$A = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$\vec{r}'(t) = A \vec{r}(t)$$

$$x'(t) = -\lambda x(t) + \lambda y(t)$$

$$y'(t) = \mu x(t) - \mu y(t)$$

$$\vec{r}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -\lambda x(t) + \lambda y(t) \\ \mu x(t) - \mu y(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \vec{r}(t)$$

Then $\vec{r}'(t) = A \vec{r}(t)$

Eigenvalues for A : $\lambda_1 = 0, \lambda_2 = -(\lambda + \mu)$

Eigenvectors for A : $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -\mu \\ \lambda \end{pmatrix}$

Since $\vec{v}_1, \vec{v}_2$ is a basis, we can always write

$$\vec{r}(t) = c(t) \vec{v}_1 + d(t) \vec{v}_2$$

$$\vec{r}'(t) = c'(t) \vec{v}_1 + d'(t) \vec{v}_2$$

$$A \vec{r}(t) = A(c(t) \vec{v}_1 + d(t) \vec{v}_2) = c(t) A \vec{v}_1 + d(t) A \vec{v}_2$$

$$= -(\lambda + \mu) d(t) \vec{v}_2$$

Since $\vec{r}' = A \vec{r}$, we get

$$c'(t) \vec{v}_1 + d'(t) \vec{v}_2 = -(\lambda + \mu) d(t) \vec{v}_2$$

Hence $c'(t) = 0, d'(t) = -(\lambda + \mu) d(t)$

$$c(t) = C, d(t) = D e^{-(\lambda + \mu)t}$$

Hence $\vec{r}(t) = c(t) \vec{v}_1 + d(t) \vec{v}_2 = C \begin{pmatrix} 1 \\ 1 \end{pmatrix} + D e^{-(\lambda + \mu)t} \begin{pmatrix} -\mu \\ \lambda \end{pmatrix}$

$$= \begin{pmatrix} C - \mu D e^{-(\lambda + \mu)t} \\ C + \lambda D e^{-(\lambda + \mu)t} \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} P_{00}(t) \\ P_{10}(t) \end{pmatrix}$$

Determine C and D : $P_{00}(0) = 1, P_{10}(0) = 0$

$$x(0) = 1, y(0) = 0$$

This means: $t=0$

Substituted

$$\begin{cases} C - \mu D = 1 \\ C + \lambda D = 0 \end{cases}$$

$$\mu D - \lambda D = 1 - 0 \Rightarrow D(\lambda + \mu) = 1$$

$$D = \frac{1}{\lambda + \mu}$$

$$C = -\lambda D = -\frac{\lambda}{\lambda + \mu}$$

$$P_{00}(t) = C - \mu D e^{-(\lambda + \mu)t} = \frac{-\lambda}{\lambda + \mu} + \frac{1}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

$$= \frac{\lambda}{\lambda + \mu} + \frac{1}{\lambda + \mu} e^{-(\lambda + \mu)t} \rightarrow \frac{\lambda}{\lambda + \mu}$$