

Limit probabilities

Σ - a cont time MC.

So assume that we for each j have a number $0 \leq p_j \leq 1$ such $\sum p_j = 1$ and

$$p_{ij}(t) \rightarrow \underbrace{p_j}_{\text{independently of } i} \text{ as } t \rightarrow \infty,$$

then $p_1, p_2, \dots$ is called a set of limit probs.

Let us see how we can find p_j (assuming they exist).

Additional condition: $p'_{ij}(t) \rightarrow \text{converges} \Rightarrow p'_{ij}(t) \rightarrow 0$

Note:
$$\begin{array}{ccc} p_{ij}(t+1) - p_{ij}(t) = p'_{ij}(c) & \text{Mean Value Theorem} & \\ \downarrow & \downarrow & \downarrow \\ p_j - p_j & = & 0 \end{array} \quad c \in (t, t+1)$$

Kolmogorov's Forward Equation:

$$p'_{ij}(t) = \sum_{k \neq j} q_{kj} p_{ik}(t) - v_j p_{ij}(t)$$

$\downarrow t \rightarrow \infty$

$$\rightarrow 0 = \sum_{k \neq j} q_{kj} p_k - v_j p_j \quad (*)$$

i.e.

$$v_j p_j = \sum_{k \neq j} q_{kj} p_k \quad (*) \text{ one equation for each } j.$$

Conditions for the limit probs to exist

- (i) Σ is irreducible; i.e. for any two states, the prob of getting from i to j (at some time) is positive
- (ii) Σ is positive recurrent in the sense that for any state i the expectation of the first return to i is finite.

Finite state space

$$S = \{1, 2, \dots, N\} \quad \left. \begin{array}{l} \text{rates } v_i \\ \text{trans probs } p_{ij} \end{array} \right\} \quad q_{ij} = v_i p_{ij}$$

Introduce the matrix:

$$Q = \begin{bmatrix} -v_1 & q_{12} & q_{13} & \dots & q_{1N} \\ q_{21} & -v_2 & q_{23} & \dots & q_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{N1} & q_{N2} & \dots & \dots & -v_N \end{bmatrix}$$

Remember that

$$\sum_j q_{ij} = v_i$$

$$-v_i + \sum_j q_{ij} = 0$$

columns are linearly

independent $\Rightarrow Q$ is singular

$$\text{Find the } -v_1 p_1 + q_{12} p_2 + \dots + q_{1N} p_N = 0$$

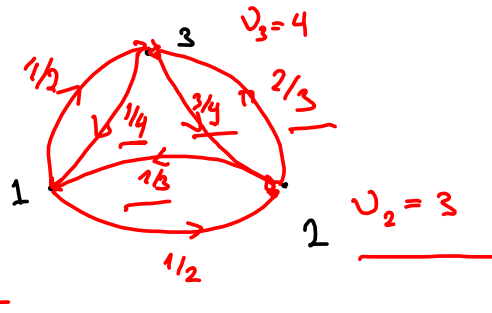
Equation (*) corresponds to

$$(p_1, \dots, p_N) Q = 0$$

Transpose: $Q^T \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix} = \underline{0} \quad \underline{p_1 + p_2 + \dots + p_N = 1}$

Why is there a nonzero solution? We need Q^T to be a singular matrix, i.e. the rows/columns must be linearly dependent.

The Perron-Frobenius Theorem guarantees that
i.e. there is a positive solution.

Example:

$$q_{12} = v_1 p_{12} = 2 \cdot \frac{1}{2} = 1, \quad q_{13} = v_1 p_{13} = 2 \cdot \frac{1}{4} = \frac{1}{2}$$

$$q_{21} = v_2 p_{21} = 3 \cdot \frac{1}{3} = 1, \quad q_{23} = v_2 p_{23} = 3 \cdot \frac{2}{3} = 2$$

$$q_{31} = v_3 p_{31} = 4 \cdot \frac{1}{4} = 1, \quad q_{32} = v_3 p_{32} = 4 \cdot \frac{3}{4} = 3$$

$$Q = \begin{bmatrix} -v_1 & q_{12} & q_{13} \\ -v_2 & & \\ & & \end{bmatrix}$$

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 3 & -4 \end{bmatrix}, \quad Q^T = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 2 & -4 \end{bmatrix}$$

Need to solve: $Q^T \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0 \Rightarrow \text{solutions}$

$$p_1 = \frac{6}{5} p_3$$

$$p_2 = \frac{7}{5} p_3$$

$$\left(\frac{6}{5} p_3, \frac{7}{5} p_3, p_3 \right)$$

Limit prob:

$$p_1 = \frac{6}{18}$$

$$p_2 = \frac{7}{18}$$

$$p_3 = \frac{5}{18}$$

$$\text{Choose } p_3 = \frac{5}{18}$$

$$\left(\frac{6}{18}, \frac{7}{18}, \frac{5}{18} \right)$$

Observation: The limit probabilities $(p_1, p_2, \dots)$ form an equilibrium state:

Argument: Start the process in the state $(p_1, p_2, \dots)$

$$P[X(t)=j] = \sum_k P[X(0)=i, X(t)=j] P_k = \sum_k P_{ik}(t) P_{kj}$$

$$= \sum_k P_{ik}(t) \lim_{s \rightarrow \infty} P_{ik}(s) \stackrel{(*)}{=} \lim_{s \rightarrow \infty} \sum_k P_{ik}(s) P_{kj}(t)$$

$$= \lim_{s \rightarrow \infty} P_{ij}(s+t) = P_{ij}$$