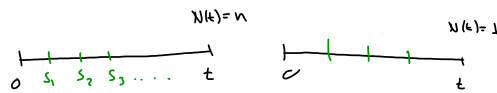


Distribution of arrival times

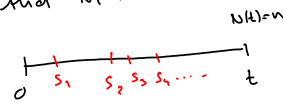
Proposition: $P[T_1 \leq s | N(t)=1] = \frac{s}{t}$ (uniform distr.)

Proof: $P[T_1 \leq s | N(t)=1] = \frac{P[T_1 \leq s \wedge N(t)=1]}{P[N(t)=1]}$

$$= \frac{P[N(s)=1 \wedge N(t)-N(s)=0]}{P[N(t)=1]}$$

$$\stackrel{\text{ind. incr.}}{=} \frac{P[N(s)=1] P[N(t)-N(s)=0]}{P[N(t)=1]} = \frac{\frac{(ts)^1}{1!} e^{-ts} \cdot \frac{(t-s)^0}{0!} e^{-(t-s)}}{\frac{(t)^1}{1!} e^{-t}} = \frac{s}{t}$$

What about the distribution of $\underbrace{s_1, s_2, \dots, s_n}_{\text{first arrival times}}$ given that $N(t)=n$.



Turns out that they are uniformly distributed:



The arrival times take values in the set

$$\Delta_t = \{(s_1, s_2, \dots, s_n) : 0 \leq s_1 \leq s_2 \leq \dots \leq s_n \leq t\}$$

What is the density function $\varphi(s_1, s_2, \dots, s_n)$?

Defn: Order statistics.

Random variable: $T_1, T_2, \dots, T_n$ independent with dist. func. f .

Let:

$T_{(1)}$ = the smallest of $T_1, \dots, T_n$

$T_{(2)}$ = the second smallest of $T_1, T_2, \dots$

...

$T_{(n)}$ = the largest of $T_1, T_2, \dots, T_n$.

Example: Three dice / blue T_1
red T_2
green T_3

Scenario 1: blue 3, red 5, green 1

$$T_{(1)} = 1, T_{(2)} = 3, T_{(3)} = 5$$

Scenario 2: blue 2, red 1, green 6

$$T_{(1)} = 1, T_{(2)} = 2, T_{(3)} = 6$$

What is the density function of $T_{(1)}, T_{(2)}, \dots, T_{(n)}$?

State space: $\Delta = \{(y_1, y_2, \dots, y_n) : y_1 \leq y_2 \leq y_3 \leq \dots \leq y_n\}$

φ is defined on Δ .

$$\varphi(y_1, y_2, \dots, y_n) = n! f(y_1) f(y_2) \dots f(y_n)$$

$$\left. \begin{array}{l} T_1 = y_1, T_2 = y_2, \dots, T_n = y_n \\ T_{i1} = y_1, T_{i2} = y_2, \dots \end{array} \right\} \begin{array}{l} n \text{ factorial way to get} \\ y_1, y_2, \dots, y_n \end{array}$$

Back to arrival times: Each T_i is uniformly distributed on $[0, t]$, i.e. it has a density

$$f(y) = \frac{1}{t} \quad \text{In this case}$$

$$\varphi(y_1, y_2, \dots, y_n) = n! \left(\frac{1}{t}\right)^n = \frac{n!}{t^n}$$

Theorem: Assume that $N(t)=n$. Then the arrival times $s_1, s_2, \dots, s_n$ are distributed as

Inhomogeneous Poisson Process

What happens if the rate λ varies with time: $\lambda(t) \geq 0$

Def: An inhomogeneous Poisson process N with rate function $\lambda(t)$ is a process N taking values in $\{0, 1, 2, 3, \dots\}$ such that:

- (i) $N(0) = 0$
- (ii) N has independent increments
- (iii) $P[N(t+h) - N(t) = 1] = \lambda(t)h + o(h)$
- (iv) $P[N(t+h) - N(t) \geq 2] = o(h)$

Assume $\lambda(t)$ is continuous:
Mean value function: $m(t) = \int_0^t \lambda(s) ds$

Proposition: $P[N(t) = 0] = e^{-m(t)}$

Proof: $y(t) = P[N(t) = 0]$

$$y(t+h) = P[N(t+h) = 0] = P[N(t) = 0 \wedge N(t+h) - N(t) = 0]$$

$$\text{ind. incr. } P[N(t) = 0] P[N(t+h) - N(t) = 0]$$

$$= y(t) (1 - P[N(t+h) - N(t) \geq 1])$$

$$= y(t) (1 - \lambda(t)h - o(h))$$

$$\frac{y(t+h) - y(t)}{h} = \frac{y(t)(1 - \lambda(t)h - o(h)) - y(t)}{h}$$

$$= \frac{-\lambda(t)y(t) - y(t)o(h)}{h} \xrightarrow{h \rightarrow 0} -\lambda(t)y(t)$$

Hence $y'(t) = -\lambda(t)y(t)$. Diff. eq.

We get

$$\frac{y'(s)}{y(s)} = -\lambda(s)$$

Integrating

$$\int_0^t \frac{y'(s)}{y(s)} ds = - \int_0^t \lambda(s) ds$$

$$\left[\ln y(s) \right]_0^t = -m(t)$$

$$\ln y(t) - \ln y(0) = -m(t)$$

$$\ln y(t) = -m(t) \Rightarrow \underline{\underline{y(t) = e^{-m(t)}}}$$

Observation: $\underbrace{P[T_1 < t]}_{F_{T_1}(t)} = P[N(t) > 0] = 1 - P[N(t) = 0]$
 $= 1 - e^{-m(t)}$

Density function of $P[T_1, t]$: $F'_{T_1}(t) = 0 - e^{-m(t)} \cdot (-\underbrace{m'(t)}_{\lambda(t)})$
 $= \lambda(t) e^{-m(t)}$

Theorem: $P[N(t) = n] = \frac{m(t)^n}{n!} e^{-m(t)}$
 i.e. $N(t)$ is Poisson distributed with rate $m(t)$.

Proof: By induction on n . Already proved for $n=0$: $P[N(t)=0] = e^{-m(t)} = \frac{m(t)^0}{0!} e^{-m(t)}$

Assume that the statement holds for n . That is for any inhom. Poisson process and any t , we have

$$P[N(t) = n] = \frac{m(t)^n}{n!} e^{-m(t)}$$

Want to prove that

$$P[N(t) = n+1] = \frac{m(t)^{n+1}}{(n+1)!} e^{-m(t)}$$

Conditioning on T_1 , we have

$$P[N(t) = n+1] = \int_0^t P[N(t) = n+1 \mid T_1 = s] \underbrace{\lambda(s) e^{-m(s)}}_{\text{density of } T_1} ds$$

$$= \int_0^t \underbrace{P[N(t) = n+1 \mid T_1 = s]}_{\text{inhom. Poisson}} \lambda(s) e^{-m(s)} ds$$

rate function $\lambda_s(t) = \lambda(t+s)$
 mean value function $m(t) = m(s)$

induction hyp = $\int_0^t \frac{(m(t)-m(s))^n}{n!} \cancel{e^{-(m(t)-m(s))}} \cdot \cancel{\lambda(s) e^{-m(s)}}$

$$= \frac{e^{-m(t)}}{n!} \int_0^t (m(t)-m(s))^n \lambda(s) ds$$

Change of variables:
 $u = m(t) - m(s)$
 $du = -\lambda(s) ds$

$$= \frac{e^{-m(t)}}{n!} \int_{m(t)}^0 u^n (-du) = \frac{e^{-m(t)}}{n!} \int_0^{m(t)} u^n du$$

$$= \frac{e^{-m(t)}}{n!} \left[\frac{u^{n+1}}{n+1} \right]_0^{m(t)} = \frac{e^{-m(t)}}{n!} \cdot \frac{m(t)^{n+1}}{(n+1)!}$$

$$= \frac{m(t)^{n+1}}{(n+1)!} e^{-m(t)} \quad \text{Huvvah!}$$

Example: A store is open from 10 to 20.

Customers arrive according to an inhomogeneous PP.
with rate function $r(t) = 3t(10-t)$, when $t=0$
corresponds to 10.

What is the distribution of customers one may
expect to get during one day?

Mean value function: $m(t) = \int_0^t r(s) ds = \int_0^t 3s(10-s) ds$
 $= 3 \int_0^t (10s - s^2) ds = 3 \left[5s^2 - \frac{s^3}{3} \right]_0^t$
 $= \underline{\underline{15t^2 - t^3}}$

At the end of the day: $m(10) = \underbrace{15 \cdot 10^2}_{1500} - \underbrace{10^3}_{1000} = \underline{\underline{500}}$

Number of customers is Poisson distributed with mean 500: