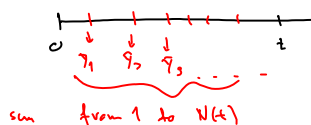


Compound Poisson process

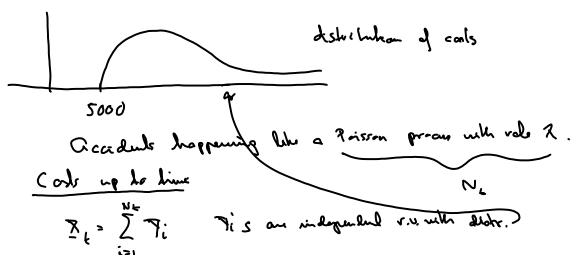
Poisson process: N with rate λ

$T_1, T_2, T_3, \dots$ are independent and ident. distr. random variables, independent of N .

Define: $X(t) = \sum_{i=1}^{N(t)} T_i$ is a compound Poisson process.



Example: Running a car insurance company



Proposition: The mean of the CPP $X_t = \sum_{i=1}^{N_t} T_i$ is given
 $E[X_t] = \lambda t E[T]$ (same distr. as all T_i)

$$\begin{aligned} \text{Proof: } E[X_t] &= E\left[\sum_{i=1}^{N_t} T_i\right] = \sum_{n=0}^{\infty} E\left[\sum_{i=1}^{N_t} T_i \mid N_t = n\right] P[N_t = n] \\ &= \sum_{n=0}^{\infty} E\left[\sum_{i=1}^n T_i \mid N_t = n\right] P[N_t = n] \\ &= \sum_{n=0}^{\infty} E\left[\sum_{i=1}^n T_i\right] P[N_t = n] = \sum_{n=0}^{\infty} n E[T] P[N_t = n] \\ &= E[T] \sum_{n=0}^{\infty} n P[N_t = n] = E[T] E[N_t] = \lambda t E[T] \end{aligned}$$

Proposition: $\text{Var}(X_t) = \lambda t E[T^2]$

Prerequisites: $E[N_t] = \lambda t$, $\text{Var}(N_t) = \lambda t$

$$\begin{aligned} \text{Var}(N_t) &= E[N_t^2] - E[N_t]^2 \Rightarrow E[N_t^2] = \text{Var}(N_t) + E[N_t]^2 = \lambda t + \lambda^2 t^2 \\ \text{Let } T_1, T_2, \dots, T_n \text{ are i.i.d.:} \\ E[(T_1 + T_2 + \dots + T_n)^2] &= E\left[\sum_{i,j} T_i T_j\right] = E\left[\sum_{i=1}^n T_i^2\right] + \sum_{i \neq j} E[T_i T_j] \\ &= n E[T^2] + n(n-1) E[T]^2 \end{aligned}$$

$$\begin{aligned} \text{Proof of proposition: } \text{Var}(X_t) &= E[X_t^2] - E[X_t]^2 \\ &= E[X_t^2] - \lambda^2 t^2 E[T]^2 \end{aligned}$$

$$\begin{aligned} \text{Need: } E[X_t^2] &= E\left[\left(\sum_{i=1}^{N_t} T_i\right)^2\right] = \sum_{n=0}^{\infty} E\left[\left(\sum_{i=1}^{N_t} T_i\right)^2 \mid N_t = n\right] P[N_t = n] \\ &= \sum_{n=0}^{\infty} E\left[\left(\sum_{i=1}^n T_i\right)^2 \mid N_t = n\right] P[N_t = n] \\ &= \sum_{n=0}^{\infty} (n E[T^2] + n(n-1) E[T]^2) P[N_t = n] \\ &= E[T^2] \sum_{n=0}^{\infty} n P[N_t = n] + E[T]^2 \sum_{n=0}^{\infty} n^2 P[N_t = n] - E[T]^2 \sum_{n=0}^{\infty} n P[N_t = n] \\ &= E[T^2] E[N_t] + E[T]^2 (E[N_t^2] - E[N_t]) - E[T]^2 E[N_t] \\ &= \lambda t E[T^2] + (\lambda t + \lambda^2 t^2) E[T]^2 - \lambda t E[T]^2 - \lambda t E[T]^2 = \lambda t E[T^2] + \lambda^2 t^2 E[T]^2 - \lambda t E[T]^2 \\ &= \lambda t E[T^2] \end{aligned}$$

Hence

$$\begin{aligned} \text{Var}(X_t) &= E[X_t^2] - \lambda^2 t^2 E[T]^2 = \lambda t E[T^2] + \lambda^2 t^2 E[T]^2 - \lambda^2 t^2 E[T]^2 \\ &= \lambda t E[T^2] \end{aligned}$$

Example: Car insurance.

$\lambda = 10$ (10 acc per week).

Cost: Average 10^4 , Variance $4 \cdot 10^8$

Expected cost over a year.

$$E[X_t] = \lambda t E[Y]$$

$$E[X_{52}] = 10 \cdot 52 \cdot 10^4 = \underline{\underline{5.2 \cdot 10^6}}$$

$$\text{Var} = \lambda t E[Y^2]$$

$$\text{Var}(X_{52}) = 10 \cdot 52 \cdot E[Y^2]$$

$$\text{Var} Y = E[Y^2] - E[Y]^2$$

$$E[Y^2] = \text{Var} Y + E[Y]^2$$

$$= 520 (4 \cdot 10^8 + 10^8) = 520 \cdot 5 \cdot 10^8 = 5.20 \cdot 5 \cdot 10^{10} = 26 \cdot 10^{10}$$

$$= \underline{\underline{2.6 \cdot 10^{11}}}$$

Next time Ch 6: Cont. time MC

