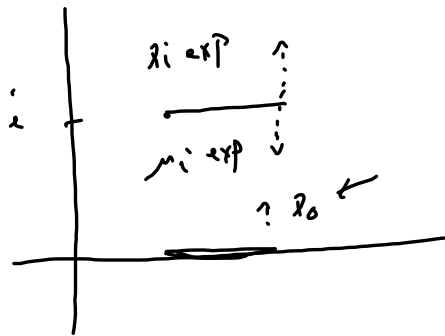


Limit probabilitiesLife and birth processLook for an equil. distr.

$$P_0, P_1, P_2, \dots$$

<u>State</u>	Goes out	Comes in
--------------	----------	----------

$$0 \quad \underline{\lambda_0 P_0 = \mu_1 P_1}$$

$$1 \quad \cancel{\lambda_1 P_1} + \cancel{\mu_1 P_1} = \cancel{\lambda_0 P_0} + \mu_2 P_2 \Rightarrow \underline{\lambda_1 P_1 = \mu_2 P_2}$$

$$2 \quad \cancel{\lambda_2 P_2} + \cancel{\mu_2 P_2} = \cancel{\lambda_1 P_1} + \mu_3 P_3 \Rightarrow \underline{\lambda_2 P_2 = \mu_3 P_3}$$

$$\vdots$$

$$\Rightarrow \lambda_n P_n = \mu_{n+1} P_{n+1}$$

$$n \quad \text{Top line: } P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0$$

$$P_3 = \frac{\lambda_2}{\mu_3} P_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} P_0$$

$$\vdots$$

$$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_0}{\mu_n \mu_{n-1} \dots \mu_1} P_0$$

$$\text{Need: } P_0 + P_1 + P_2 + \dots + P_n + \dots = 1$$

$$P_0 + \sum_{n=1}^{\infty} P_n = P_0 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_0}{\mu_n \mu_{n-1} \dots \mu_1} P_0 = 1$$

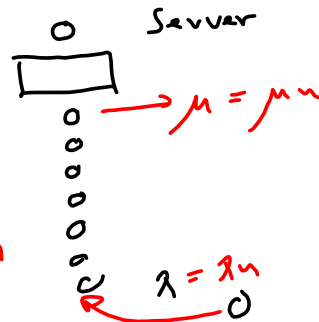
$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \dots \lambda_0}{\mu_n \dots \mu_1}}$$

Result: Limit prob exist if and only if $\sum \frac{\lambda_{n-1} \dots \lambda_0}{\mu_n \dots \mu_1}$ converges. And then P_0 and P_n

use this
to converg.

Example:

$$\text{Hence } \frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \dots \mu_1} \\ = \frac{\lambda^n}{\mu^n} = \left(\frac{\lambda}{\mu}\right)^n$$



Thus: $\sum \frac{\lambda_{n-1} \dots \lambda_0}{\mu_n \dots \mu_1} = \sum \left(\frac{\lambda}{\mu}\right)^n$ geometric converging $\lambda < \mu$.

$$\text{Hence } P_0 = 1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = \frac{1}{1 - \frac{\lambda}{\mu}} = \frac{\mu}{\mu - \lambda}$$

$$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \dots \mu_1} P_0 = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{\lambda}{\mu}\right)^n \frac{\mu}{\mu - \lambda}$$

Example: $\mu_n = n\mu$, $\lambda_n = n\lambda$ for $n \geq 1$. $\lambda_0 > 0$.

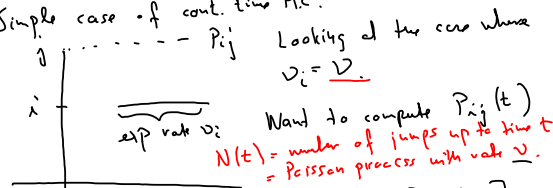
$$\text{Crucial product: } \frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_2 \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \dots \mu_2 \mu_1} = \frac{\cancel{(n-1)\lambda} \cancel{(n-2)\lambda} \dots \cancel{(2\lambda)} \cancel{(1\lambda)} \lambda}{n\mu \cancel{(n-1)\mu} \dots \cancel{(2\mu)} \mu}$$

$$= \frac{\lambda_0}{n} \frac{\lambda^n}{\mu^n}$$

When do we have convergence of $\sum \frac{\lambda_0}{n} \frac{\lambda^n}{\mu^n} = \lambda_0 \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n}$
 (converges for $\left(\frac{\lambda}{\mu}\right) < 1$; i.e. $\lambda < \mu$.)

Uniformization

Simple case of cont. time MC.

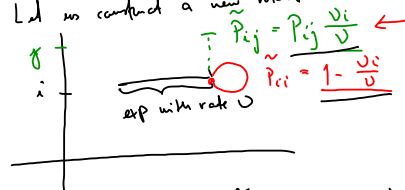


$$P_{ij}(t) = P[\tilde{X}(t) = j | \tilde{X}(0) = i] = P_i[\tilde{X}(t) = j]$$

$$= \sum_{n=0}^{\infty} P_i[\tilde{X}(t) = j | N(t) = n] P[N(t) = n]$$

$$= \sum_{n=0}^{\infty} (P^n)_{ij} \frac{(vt)^n}{n!} e^{-vt}$$

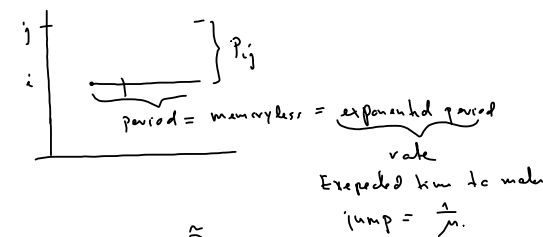
Extended case: Assuming that the v_i 's are different but bounded: $v_i \leq v$ for all i .
Let us construct a new Markov chain $\tilde{X}$ in this way



Consistency check: $\sum_j \tilde{P}_{ij} = \sum_{j \neq i} P_{ij} \frac{v_i}{v} + 1 - \frac{v_i}{v}$

$$= \frac{v_i}{v} \left(\sum_{j \neq i} P_{ij} \right) + 1 - \frac{v_i}{v} = 1$$

What does $\tilde{X}$ look like from the outside?



Expected time to $\tilde{X}$ makes a jump:

prob of making a jump after the first period is $\frac{v_i}{v}$

$$\frac{1}{v} \frac{v_i}{v} + \frac{2}{v} (1 - \frac{v_i}{v}) \frac{v_i}{v} + \frac{3}{v} (1 - \frac{v_i}{v})^2 \frac{v_i}{v}$$

$$+ \dots + \frac{n}{v} (1 - \frac{v_i}{v})^{n-1} \frac{v_i}{v} + \dots$$

$$= \frac{v_i}{v^2} \sum_{n=1}^{\infty} n (1 - \frac{v_i}{v})^{n-1}$$

$$= \frac{v_i}{v^2} \frac{1}{(1 - (1 - \frac{v_i}{v}))^2}$$

$$= \frac{v_i}{v^2} \frac{1}{\frac{v_i}{v}} = \frac{1}{v_i}$$

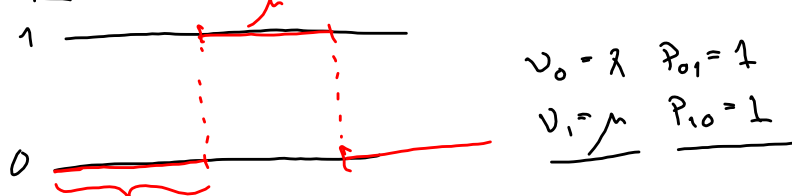
Diff: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
 $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$

The expected time $\frac{1}{v_i}$, the rate is of the exponential

This shows that $\tilde{X}$ is a cont. time MC with transition prob $\tilde{P}_{ij}$ and rate v_i (just like X).
 $v_i \leq v$

By relabeling similar to the case we started with

Example: Machine with two states: 0 working, 1 broken



Want to compute: P_{00}^t = machine working at time t when started in 0.

Uniformization: $\nu = \lambda + \mu$

$$\tilde{p}_{01} = p_{01} \frac{\nu_0}{\nu} = 1 \cdot \frac{\lambda}{\lambda + \mu} = \frac{\lambda}{\lambda + \mu}$$

$$\tilde{p}_{00} = 1 - \frac{\nu_0}{\nu} = 1 - \frac{\lambda}{\lambda + \mu} = \frac{\mu}{\lambda + \mu}$$

$$\tilde{p}_{10} = p_{10} \frac{\nu_1}{\nu} = 1 \cdot \frac{\mu}{\lambda + \mu} = \frac{\mu}{\lambda + \mu}$$

$$\tilde{p}_{11} = 1 - \frac{\nu_1}{\nu} = 1 - \frac{\mu}{\lambda + \mu} = \frac{\lambda}{\lambda + \mu}$$

Calculation

$$P_{00}^t = \sum_{n=0}^{\infty} (\tilde{P}^n)_{00} \frac{(\nu t)^n}{n!} e^{-\nu t} = e^{-(\lambda + \mu)t} + \sum_{n=1}^{\infty} \frac{\mu}{\lambda + \mu} \frac{[(\lambda + \mu)t]^n}{n!}$$

$$= e^{-(\lambda + \mu)t} + \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t} \left(1 + \sum_{n=1}^{\infty} \frac{[(\lambda + \mu)t]^n}{n!} - 1 \right)$$

$$= e^{-(\lambda + \mu)t} + \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t} \left[e^{(\lambda + \mu)t} - 1 \right]$$

$$= e^{-(\lambda + \mu)t} + \frac{\mu}{\lambda + \mu} \left[1 - e^{-(\lambda + \mu)t} \right]$$

$$= \frac{\mu}{\lambda + \mu} + \frac{\lambda + \mu}{\lambda + \mu} e^{-(\lambda + \mu)t} - \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

$$= \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

xxx-ndi

$$e^{tR} = \lim_{n \rightarrow \infty} \left(I + \frac{tR}{n} \right)^n$$