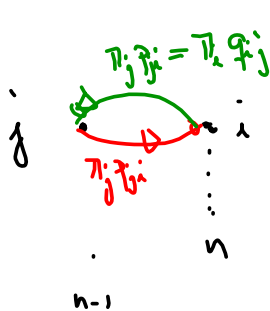


Time reversible Markov chains

Assume $\bar{X}$ a irreducible Markov chain with stationary distribution π



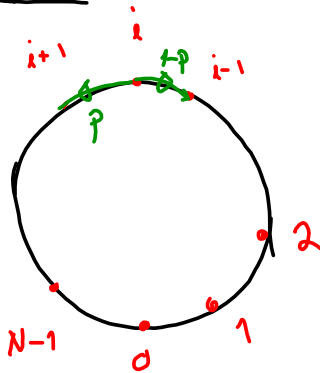
Backwards transition matrix Q :

$$q_{ij} = \frac{\pi_j p_{ji}}{\pi_i}$$

Check consistency:

$$\sum_j q_{ij} = \sum_j \frac{\pi_j p_{ji}}{\pi_i} = \frac{1}{\pi_i} \left(\sum_j \pi_j p_{ji} \right) = 1$$

Example: Random walks on a circle.



Stationary distribution: $\pi_i = \frac{1}{N}$

What is the time reversed probabilities:

$$q_{ij} = \frac{\pi_j p_{ji}}{\pi_i} = p_{ji}$$

Since $\pi_j = \pi_i = \frac{1}{N}$

$$q_{i,i+1} = p_{i+1,i} = 1-p$$

$$q_{i,i-1} = p_{i-1,i} = p$$

$$\underbrace{q_{i,i+1}}_{1-p} \neq \underbrace{p_{i+1,i}}_p \text{ unless } p = \frac{1}{2}.$$

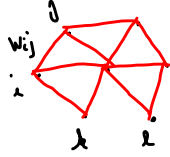
Definition: $\bar{X}_n$ is time reversible if $q_{ij} = p_{ij}$ for all i, j , i.e.

$$p_{ij} = \frac{\pi_j p_{ji}}{\pi_i}$$

i.e.

$$\boxed{\pi_i p_{ij} = \pi_j p_{ji}}$$

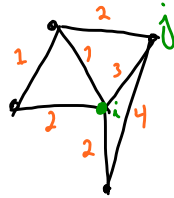
Example: Random walks on weighted graphs



Points (vertices) connected by lines (edges)

A weight $w_{ij} = w_{ji}$ is a positive number.

$w_{kl} = 0$ when there is no edge between k and l .



Induced Markov chain:



$$p_{ij} = \frac{w_{ij}}{\sum_k w_{ik}}$$

Consistency check:

$$\sum_j p_{ij} = \sum_j \frac{w_{ij}}{\sum_k w_{ik}} = \frac{1}{\sum_k w_{ik}} \sum_j w_{ij} = 1$$

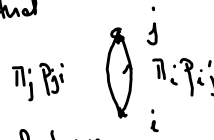
$$p_{ij} = \frac{3}{8}$$

Claim: This Markov chain is always time reversible.

How to find a distribution π such that

$$\pi_i p_{ij} = \pi_j p_{ji}$$

detailed balance equation.



$$\sum \pi_i = 1.$$

We get: $\pi_j = \frac{\pi_i p_{ij}}{p_{ji}}$

$$\left(\frac{1}{\sum_j} \right) \sum_j \pi_j = \sum_j \frac{\pi_i p_{ij}}{p_{ji}} = \pi_i \sum_j \frac{p_{ij}}{p_{ji}} =$$

$$= \pi_i \sum_j \frac{\frac{w_{ij}}{\sum_k w_{ik}}}{\frac{w_{ji}}{\sum_k w_{jk}}} = \pi_i \sum_j \frac{\sum_k w_{jk}}{\sum_k w_{ik}} = \left(\frac{\pi_i}{\sum_k w_{ik}} \right) \sum_j \sum_k w_{jk}$$

$$\text{Hence } \pi_i = \frac{\sum_k w_{ik}}{\sum_j \sum_k w_{jk}}$$

invariant/stationary distribution

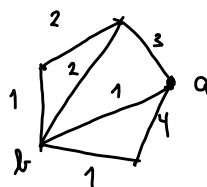
What does this mean?

$$\pi_i = \frac{\sum_k w_{ik}}{\sum_j \sum_k w_{jk}}$$

the sum of all weights "going out" from i

the total weight on all edges multiplied by two.

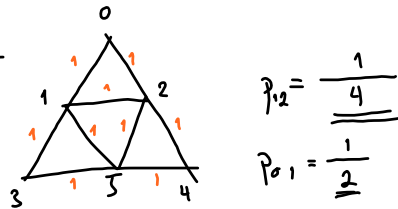
Example:



$$\pi_a = \frac{3+1+1}{2 \cdot (14)} = \frac{5}{14} = \frac{5}{14}$$

$$p_{ab} = \frac{1}{5}$$

Example:



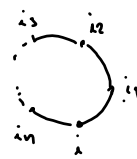
$$p_{12} = \frac{1}{4}$$

$$p_{01} = \frac{1}{2}$$

$$\pi_1 = \frac{4}{2 \cdot 9} = \frac{2}{9}$$

$$\pi_3 = \frac{2}{2 \cdot 9} = \frac{1}{9}$$

Question: Is there a criterion for when a MC is time reversible?



if time reversible

$$\begin{aligned} & \pi_{i1} p_{i1i2} p_{i2i3} \dots p_{i4i1} \\ &= \pi_{i2} p_{i2i1} p_{i1i3} \dots p_{i4i2} \end{aligned}$$

Theorem: Assume that Σ is an irreducible and positive recurrent Markov chain. Then Σ is time reversible if and only if

$$p_{i1i2} p_{i2i3} \dots p_{in-1in} = p_{in} p_{in-1} \dots p_{i2i1}$$

holds for all cycles $i \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{n-1} \rightarrow i$.

"Proof": Fix two states i, j and look at all cycles

$$i \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{n-1} \rightarrow j \rightarrow i$$

Then for all $i, i_1, i_2, \dots, i_{n-1}$, we have

$$p_{i1i2} p_{i2i3} \dots p_{in-1in} = p_{ij} p_{jin} \dots p_{xi}$$

Sum over all possible $i_1, i_2, \dots, i_{n-1}$:

$$\left(\sum_{i_1, i_2, \dots, i_{n-1}} p_{i1i2} p_{i2i3} \dots p_{in-1in} \right) p_{ji} = p_{ij} \sum_{i_1, i_2, \dots, i_{n-1}} p_{jin} p_{in-1i_{n-1}} \dots p_{xi}$$

$$p_{ij}^{(n+1)} p_{ji} = p_{ij} p_{ji}^{(n+1)}$$

hence

$$\begin{aligned} \frac{\sum_{k=1}^m p_{ij}^{(k)}}{m} p_{ji} &= p_{ij} \frac{\sum_{k=1}^m p_{ji}^{(k)}}{m} \Rightarrow \pi_j p_{ji} = \pi_i p_{ij} \\ &\downarrow \text{limiting prob.} \quad \downarrow \pi_i \quad \text{detailed balance equation} \\ &\pi_j \quad \pi_i \quad \downarrow \\ &\Sigma \text{ is time reversible.} \end{aligned}$$

Strategy for proving that a MC is not time reversible:

Find three states i, j, k such that $i \rightarrow j \rightarrow k \rightarrow i$

$$p_{ij} p_{jk} p_{ki} \neq p_{ik} p_{kj} p_{ji}$$

Theorem: Assume that $\mathcal{X}$ is an irreducible MC with transition matrix P . Assume we can find a transition matrix Q and a distribution π such that

$$\pi_i P_{ij} = \pi_j Q_{ji} \quad \text{for all } i, j.$$

Then Q is the transition matrix for the time reversed chain and π is the stationary distribution for P (and also for Q).

Example (of one more reversible chain): Is this line reversible (and what is π)?



Detailed balance equation:

$$\begin{aligned} \pi_{i-1} \alpha_{i-1} &= \pi_i (1 - \alpha_i) \\ \pi_0 &= \pi_1 (1 - \alpha_1) \Rightarrow \pi_1 = \frac{\pi_0}{1 - \alpha_1} \\ \pi_1 \alpha_1 &= \pi_2 (1 - \alpha_2) \Rightarrow \pi_2 = \frac{\pi_1 \alpha_1}{(1 - \alpha_2)} = \frac{\pi_0 \alpha_1}{(1 - \alpha_1)(1 - \alpha_2)} \end{aligned}$$

In general:

$$\pi_i = \frac{\pi_0 \alpha_1 \alpha_2 \dots \alpha_{i-1}}{(1 - \alpha_1)(1 - \alpha_2) \dots (1 - \alpha_i)}$$

$$1 = \sum \pi_i = \pi_0 \sum_{i=0}^n \frac{\alpha_1 \alpha_2 \dots \alpha_{i-1}}{(1 - \alpha_1)(1 - \alpha_2) \dots (1 - \alpha_i)}$$

$$\Rightarrow \pi_0 = \frac{1}{\sum \dots}$$