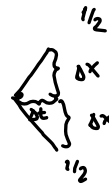
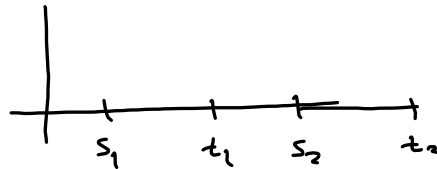


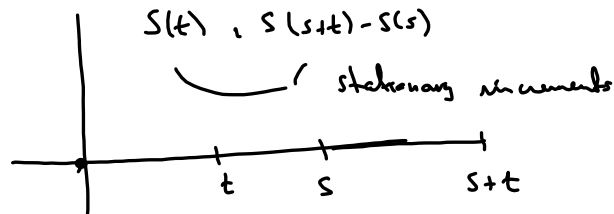
Brownian motionRandom walks

$$S(t) : E[S(t)] = 0, E[\text{Var}(S(t))] = \frac{\Delta x^2}{\Delta t} \cdot t$$

$$\Delta x^2 = \sigma^2 \Delta t, \Delta x = \sigma \sqrt{\Delta t}$$

Properties of S:

S has independent increments $S(t_2) - S(t_1)$ is independent of $S(t_1) - S(t_0)$



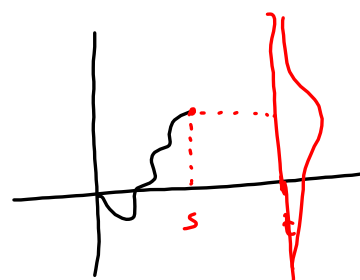
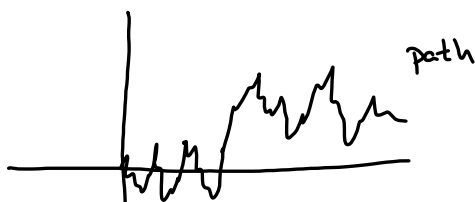
$S(t) - S(s)$ almost normally distributed with mean 0 and variance $\sigma^2(t-s)$

Definition: A Brownian motion is a stochastic process

$\{B_t\}_{t \in [0, \infty)}$ such that

- (i) $B(0) = 0$
- (ii) B has independent and stationary increments
- (iii) $B(t)$ is $N(0, \sigma^2 t)$ -distributed. (if $\sigma^2 = 1$, B is called standard)
- (iv) The paths of B are continuous.

Brownian motion, observed from now on).

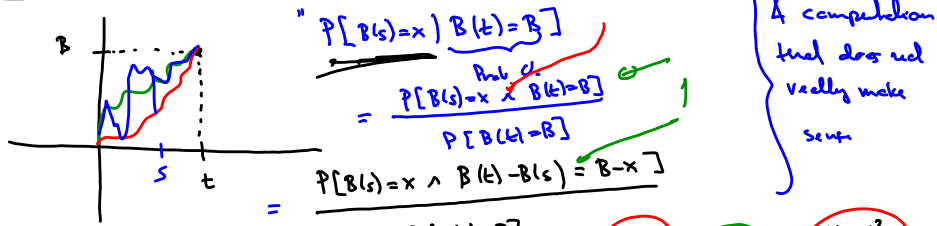


Same calculations: What is the distribution $X = B(s) + B(t)$, $t > s$.

We have: $X = B(s) + \underbrace{B(t) - B(s)}_{B(s) - B(s)}$ $= \underbrace{2B(s)}_{\text{Gaussian, independent}} + \underbrace{B(t) - B(s)}_{\text{Gaussian, mean 0}}$

Variance
 $= \text{Variance}(2B(s))$
 $+ \text{Variance}(B(t) - B(s))$
 $= 4 \text{Variance } B(s)$
 $+ \text{Variance}(B(t) - B(s))$
 $= 4s + t - s = 3s - t$

Example: (Brownian bridge) Intuition



$$\begin{aligned}
 &P[B(s)=x \mid B(t)=B] \\
 &= \frac{P[B(s)=x \wedge B(t)=B]}{P[B(t)=B]} \\
 &= \frac{P[B(s)=x \wedge B(t)-B(s)=B-x]}{P[B(t)=B]} \\
 &= \frac{P[B(s)=x] P[B(t)-B(s)=B-x]}{P[B(t)=B]} \\
 &= \frac{\frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} \cdot \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(B-x)^2}{2(t-s)}}}{\frac{1}{\sqrt{2\pi t}} e^{-\frac{B^2}{2t}}} \\
 &= K_1 e^{-\left[\frac{x^2}{2s} + \frac{(B-x)^2}{2(t-s)}\right]} \\
 &= K_1 e^{-\left[\frac{x^2}{2s} + \frac{B^2 - 2Bx + x^2}{2(t-s)}\right]} \\
 &= K_1 e^{-\left[\frac{(t-s)x^2 + B^2 - 2Bsx + x^2}{2s(t-s)}\right]} \\
 &= K_1 e^{-\left[\frac{\frac{t}{s}x^2 - 2Bsx + B^2}{2s(1-\frac{s}{t})}\right]} \\
 &= K_2 e^{-\left[\frac{x^2 - 2B\frac{s}{t}x + B^2\frac{s^2}{t^2} - B^2\frac{s^2}{t^2}}{2s(1-\frac{s}{t})}\right]} \\
 &= K_2 e^{-\frac{(x - B\frac{s}{t})^2}{2s(1-\frac{s}{t})}} \\
 &= K_3 e^{-\frac{(x - \frac{s}{t}B)^2}{2s(1-\frac{s}{t})}}
 \end{aligned}$$

this is a density
 $N(\frac{s}{t}B, s(1-\frac{s}{t}))$

