

Question: S $\left\{ \begin{array}{l} \text{finite} \\ \text{countable} \end{array} \right.$ For each $i \in S$, we have $b_i > 0$
such that $\sum_{i \in S} b_i = B < \infty$.

Then $\pi_i = \frac{b_i}{B}$ and $\sum \pi_i = 1$.

Is there a time-reversible and recurrent MC X on S having π as its stationary distribution.

Answer: Sure, in many ways. We can modify any irreducible

MC T on S to find such an X

$\downarrow$
 Q

$\downarrow$
 P

Fix $i \in S$.

$$P_{ii} = Q_{ii} + \sum_{j \neq i} (1 - \alpha_{ij}) Q_{ij}$$

$$P_{ij} = \alpha_{ij} Q_{ij} \quad \text{when } 0 < \alpha_{ij} \leq 1.$$

Consistency check

$$\sum_i P_{ij} = \sum_i Q_{ij} + \sum_{i \neq j} (1 - \alpha_{ij}) Q_{ij} \\ + \sum_{j \neq i} \alpha_{ij} Q_{ij} = \sum_j Q_{ij} = 1$$

Want to choose α_{ij} such that X is time-reversible and π is the stationary measure for X

Detailed balance equation:

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \text{for } i \neq j$$

i.e.

$$\pi_i \alpha_{ij} Q_{ij} = \pi_j \alpha_{ji} Q_{ji}$$

$$\frac{b_i}{B} \alpha_{ij} Q_{ij} = \frac{b_j}{B} \alpha_{ji} Q_{ji}$$

$$b_i \alpha_{ij} Q_{ij} = b_j \alpha_{ji} Q_{ji} \quad (*)$$

Note that $\frac{b_j Q_{ji}}{b_i Q_{ij}}$ and $\frac{b_i Q_{ij}}{b_j Q_{ji}}$ are reciprocals

and hence if one is larger than 1, the other one is smaller than 1.

Choose
$$\alpha_{ij} = \min \left\{ \frac{b_j Q_{ji}}{b_i Q_{ij}}, 1 \right\}.$$

Then

$$\alpha_{ji} = \min \left\{ \frac{b_i Q_{ij}}{b_j Q_{ji}}, 1 \right\}$$

Let us check that (*) is satisfied:

Assume first that $\frac{b_j Q_{ji}}{b_i Q_{ij}} \geq 1$. Then $\alpha_{ij} = 1$, $\alpha_{ji} = \frac{b_i Q_{ij}}{b_j Q_{ji}}$

Check (*): LHS $b_i \alpha_{ij} Q_{ij} = b_i Q_{ij}$
RHS: $b_j \alpha_{ji} Q_{ji} = \frac{b_j}{b_j} \frac{b_i Q_{ij}}{b_j Q_{ji}} Q_{ji} = b_i Q_{ij}$

Assume next that $\frac{b_j Q_{ji}}{b_i Q_{ij}} < 1$. Then

$\frac{b_i Q_{ij}}{b_j Q_{ji}} > 1$, and we are back to the same situation with i, j interchanged

In either case the detailed balance equation is satisfied and hence X (the MC generated?) is time-reversible and

has invariant measure distribution π stationary.

Applications: To Monte Carlo methods, Hastings-Metropolis algorithm

Starting point: We have a random variable U taking values in S with a distribution proportional to b_i .

$$P[U=i] = C b_i = \pi_i$$

We want to estimate

$$E[f(U)]$$

Using the methods above we first constructed a finite reversible and recurrent MC $\mathbb{X}$ with stationary distribution π_i .

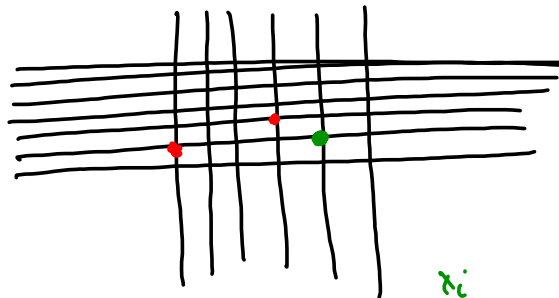
Now run the Markov Chain

$$E[f(U)] = \lim_{M \rightarrow \infty} \frac{\sum_{n=1}^M f(\mathbb{X}_n)}{M}$$

Gibbs sampler

Random vector $\vec{X}$ = $(X_1, X_2, \dots, X_n) \in \mathbb{R}^n$

$\mathbb{X}$ takes values in a discrete set S :



$$P[\vec{X} = \vec{x}] = p(\vec{x})$$

What should Q be?

$$Q(\vec{x}, \vec{y})$$

$$\vec{x} = (x_1, x_2, \dots, x_n) \quad \text{Pick } i = \{1, \dots, n\}$$

$$\vec{y} = (x_1, x_2, \dots, y_i, \dots, x_n) \leftarrow \text{according to } p(y_i)$$

$$\alpha(\vec{x}, \vec{y}) = 1,$$