Modeling by Stochastic Processes (STK 2130)
Exercises 10

Problem 1 Once more consider Exercises 8, Problem 2: A Markov chain model for the growth and replacement of trees assumes that there are three stages of growth based on the size of the tree: young tree, mature tree, and old tree. When an old tree dies, it is replaced by a young tree with probability 0.3. Order the states numerically, 1; 2 and 3. State 1 is a young tree, state 2 is a mature tree, and state 3 is an old tree. A Markov chain model for the transitions between each of the states over a period of eight years has the following transition matrix:

$$P = \begin{pmatrix} 0.25 & 0.75 & 0 \\ 0 & 0.5 & 0.5 \\ 0.3 & 0 & 0.7 \end{pmatrix}.$$

Transitions occur over an eight-year period. For example, after a period of eight years the probability that a young tree becomes a mature tree is 0.75 and the probability that it remains a young tree is 0.25. Denote the states of the tree by the Markov chain $X_n, n \geq 0$.

(i) What is the long-run proportion of time spent by the tree in each state ?
(ii) What is the long-time average state of the tree ? Or more precisely, what is

$$\lim_{n \longrightarrow \infty} \frac{X_0 + X_1 + ... + X_{n-1}}{n} \ ?$$

See Section Section 3.8 in the manuscript .

Problem 2 Consider the following (rounded) Danish fireninsurance data collected at Copenhagen Reinsurance in the period from 02/09/1987 until 02/22/1987:

Date	X_n loss in DKM
02/10/1987	4
02/11/1987	1
02/12/1987	2
02/13/1987	1
02/15/1987	5
02/16/1987	3
02/18/1987	2
02/19/1987	2
02/21/1987	1
02/22/1987	1

The above data refer to fire losses expressed in millions of Danish Krone. Assume that the claim size X_n at time $n \geq 0$ is modelled by a Markov chain with state space $I = \{0, 1, 2, 3...\}$ and transition matrix $P = (p_{ij})_{i,j \in I}$. Here $n = 0$ corresponds to 02/10/1987.

Calculate the maximum likelihood estimator $\widehat{p}_{ij}$ of the transition probabilities p_{ij} for all observed states.

Hint: See Section 3.8 in the manuscript or p. 56 in Norris.

Problem 3 Consider the Markov chain of Exercises 9, Problem 2 with state space $I = \{0, 1, 2, 3, 4, 5\}$ and transition matrix P given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find the "variance" of the Markov chain, that is

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (X_k - \mu)^2,$$

where

$$\mu \stackrel{def}{=} \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} X_k.$$