Modeling by Stochastic Processes (STK 2130)
Exercises 12

Problem 1 Consider a branching process $X_n, n \geq 0$ where the offspring distribution N has mean μ and variance σ^2.

(i) Show that

$$Var[X_2] = \begin{cases} 2\sigma^2 & \text{if } \mu = 1 \\ \frac{\sigma^2(\mu^2-1)\mu}{\mu-1} & \text{if } \mu \neq 1 \end{cases}.$$

(ii) Suppose that N is described by the offspring distribution of Exercises 11, Problem 2, that is

$$N = \sum_{i=1}^{Z} B_i,$$

where Z is Poisson distributed with parameter $\lambda = 10$ and where $B_i, i \geq 1$ are independent and binomially distributed random variables with parameters $n = 1$ and $p = 20$ %.

Compute

$$Var[X_2].$$

Hint: Use the fact that

$$Var[X] = \mathcal{P}_X^{''}(1) + \mathcal{P}_X^{'}(1) - \left(\mathcal{P}_X^{'}(1)\right)^2,$$

where $\mathcal{P}_X(t)$ is the probability generating function of a random variable X. See Section 4.1 in the manuscript (see webside).

Problem 2 (Restocking a warehouse) Consider a warehouse with a capacity of c units of stock. Denote by D_n the demand for the stock in units at time period n. Then X_n stands for the residual stock at the end of period n. The mananger of the warehouse restocks to capacity c at the beginning of period $n+1$, if

$$X_n \leq m,$$

where $m \in \{0, 1, ..., c\}$ is a certain threshold. So the dynamics of the residual stock is given by

$$X_{n+1} = \begin{cases} (c - D_{n+1})^+ & \text{if } X_n \leq m \\ (X_n - D_{n+1})^+ & \text{if } m < X_n \leq c \end{cases},$$

where $(a)^+ \stackrel{def}{=} \max(a, 0)$ for $a \in \mathbb{R}$. Suppose that $D_n, n \geq 1$ is $i.i.d$ with commmon law D given by a Poisson distribution with parameter $\lambda = 2$. Further assume that $c = 2$.

(i) Find the optimal restocking strategy of the warehouse manager by mimimizing the (lost) profit of the long-run proportion of unmet demand and the costs of long-rung frequency of restocking with respect to the threshold m, that is minimize

$$a\overline{g}(m) + b\overline{f}(m),$$

with respect to m, where $\overline{g}(m)$ is the long-run frequency of restocking, $\overline{f}(m)$ the long-run proportion of unmet demand and $a = 1$ \$. Here $a \geq 0$ is the cost for restocking and $b \geq 0$ the (lost) profit per unit stock.

(ii) Determine the optimal strategy in (i) for arbitrary a and b.

Hint: See Section 4.2 in the manuscript.

Problem 3 (Poisson process) A stochastic process $N_t \in I = \{0, 1, 2, ...\}$ is called *Poisson process with intensity* $\lambda > 0$ if

(i) $N(0) = 0$ with probability 1.

(ii) N has *independent increments*, that is the r.v.'s

$$N_{t_1} - N_{t_0}, ..., N_{t_n} - N_{t_{n-1}}$$

are independent for all $0 = t_0 < t_1 < ... < t_n, n \in \mathbb{N}$.

(iii) N has *stationary distribution*, that is the r.v.

$$N_{s+h} - N_s$$

is Poisson distributed with parameter $\lambda \cdot h$ for all $h \geq 0, s > 0$.

Show that $N_t, t \geq 0$ is a *continuous-time Markov chain* in the sense that

$$P(N_{t_{n+1}} = i_{n+1} \,|N_{t_0} = i_0, ..., N_{t_n} = i_n) = P(N_{t_{n+1}} = i_{n+1} \,|N_{t_n} = i_n)$$

for all $0 \leq t_0 \leq ... \leq t_{n+1}$ and $i_0, ..., i_{n+1} \in I$.

Problem 4 (Memoryless property) Let T be an exponentially distributed random variable.

Show that T has the memoryless property, that is

$$P(T > t + s \,|T > s) = P(T > t)$$

for all $s, t \geq 0$.

Problem 5 Consider a right-continuous Markov process $X_t, t \geq 0$ with values in a finite set I. Assume that the generator matrix (or Q-matrix) of $X_t, t \geq 0$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 3 & -3 \end{pmatrix}.$$

Use Kolmogorov's forward equation (see Remark 5.2.6 in the manuscript or Section 6.8 in Ross) to compute the transition probabilities

$$p_{ij}(t) = P(X_t = j \,|X_t = i)$$

for $i, j \in I, t \geq 0$.

Problem 6 (Cramér-Lundberg model) Consider the Danish fire insurance data collected at Copenhagen Reinsurance in the period from 02/09/1987 until 02/22/1987:

Date	Loss in DKM
02/10/1987	4.128015
02/11/1987	1.076994
02/12/1987	1.502783
02/13/1987	1.021336
02/15/1987	4.500000
02/16/1987	3.107607
02/18/1987	2.254174
02/19/1987	1.757885
02/21/1987	1.398887
02/22/1987	1.252319

The above data comprise fire losses expressed in millions of Danish Krone. Assume that the claim sizes and claim arrival times are described by the Cramér-Lundberg model (see Section 5.3 in the manuscript).

Calculate the maximum likelihood estimator of the (jump) intensity λ of the claim numbers and determine the distribution, the mean and the variance of the arrival times of the fire losses.

Hint: The likelihood function $L(\lambda)$ is given by the joint density of the holding times $S_1, ..., S_n$ of the Poisson process which models the claim numbers $N(t)$. Here, the holding times are the observed *inter-arrival times*, that is $S_i = T_i - T_{i-1}, i = 1, ..., n$, where T_i is the i-th arrival time of a claim (n sample size).