Modeling by Stochastic Processes (STK 2130)
Exercises 3

Problem 1 An insurance company classifies its car insurance policyholders in the categories "high", "intermediate", or "low" risk. In any given year, a policyholder has no accidents with probability 0.6, one accident with probability 0.2, two accidents with probability 0.1, and more than two accidents with probability 0.1. If you have no accidents, you are moved down one risk category, if you have one, you stay where you are, if you have two accidents, you move up one category, and if you have more than two, you always move to high risk.

(i) Describe the sequence of moves between categories of a policyholder as a Markov chain.

(ii) What is the probability that a policyholder who belongs to the low risk category in the third year of the contract moves to high risk three years later ?

Problem 2 (Mutation of a virus) Denote by $1, ..., N$ the N different strains of a virus. Consider a virus, which belongs to strain 1 in generation $n = 0$, i.e. $X_0 = 1$. Suppose that the virus mutates from strain i in generation n to another strain j in generation $n + 1$ with probability

$$p_{ij} = \frac{\alpha}{N-1}, \, i \neq j,$$

where $0 \leq \alpha \leq 1$. The probability that the virus doesn't mutate is given by

$$p_{ii} = 1 - \alpha.$$

Assume that the strain X_n of the virus in generation $n \geq 0$ is modelled by a Markov chain with initial strain 1.

(i) Describe the communicating classes of the Markov chain.

(ii) Find the probability that the virus of the third generation belongs to its initial strain for $\alpha = 0.05$ and $N = 10$.

Hint: See Example 3.6 in the manuscript.

Problem 3 Consider the Markov chain $(X_n)_{n\geq 0}$ of Problem 2. Define the process $\left(\widetilde{X}_n\right)_{n\geq 0}$ by

$$\widetilde{X}_n = \begin{cases} 0 & , \text{if } X_n = 1 \\ 1 & \text{if } X_n \in \{2, ..., N\} \end{cases}, n \geq 0.$$

(i) Show that $\left(\widetilde{X}_n\right)_{n\geq 0}$ is a Markov chain (with $I = \{0, 1\}$) and determine its transition probabilities.

(ii) Exploit the connection between the 2-state Markov chain $\left(\widetilde{X}_n\right)_{n\geq 0}$ and the $N-$state Markov chain $(X_n)_{n\geq 0}$ to compute the probability that the virus in Problem 2 mutates to its initial strain 1 in generation $n = 100$ for $= 0.05$ and $N = 20$.

Hint: Use Example 3.1.7 in the manuscript.