Modeling by Stochastic Processes (STK 2130)
Exercises 6

Problem 1 Let the Markov chain $X_n, n \geq 0$ consisting of the states $0, 1, 2, 3$ have the transition matrix

$$P = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Determine which states are transient and which are (positive) recurrent.

Hint: Describe the Markov chain by a diagram and use Theorem 3.4.4 in the manuscript

Problem 2 Let S_n be the price of a stock at time n (in weeks) such that

$$S_n = 0\$ \text{ or } 1\$ \text{ or } 2\$, \ n = 0, 1, 2, ...$$

Suppose that the stock prices $(S_n)_{n\geq 0}$ are modelled by a Markov chain with initial distribution $P(S_0 = 1) = 1$ and transition matrix P given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.6 & 0.35 \end{pmatrix}.$$

What is the probability that the time between two consecutive visits of the stock price to 1\$ is strictly less than 3 weeks ? Or more precisely: Compute the probability

$$P_i\left(S_i^{(r)} \leq 2 \text{ weeks} \,\middle|\, T_i^{(r-1)} < \infty\right)$$

for $i = 1$ and $r \geq 1$, where $T_i^{(r)}$ is the $r-$th passage time and $S_i^{(r)}$ the length of the $r-$th excursion to i.

Hint: Use the formula of Lemma 3.4.3 in the manuscript (webpage). Or see Example 3.4.5 (car insurance) in the manuscript.

Problem 3 (Virus mutation revisited) Suppose a virus can exist in N different strains and in each generation either stays the same, or with probability $0 < \alpha < 1$ mutates to another strain, which is chosen at random. Denote by Y_n the state that either the virus belongs to the initial strain or to another in generation $n \geq 0$. As it was shown in Problem 3, Exercises 3 $Y_n, n \geq 0$ can be modelled by a Markov chain with state space $I = \{0, 1\}$ and transition matrix

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} 1-\alpha & \alpha \\ \alpha/(N-1) & 1-\alpha/(N-1) \end{pmatrix}.$$

Here 0 stands for the initial strain, whereas 1 means another strain.

Show that all states of this chain are recurrent.

Hint: Check that $p_{00}^{(n)} = p_{01}(p_{11})^{n-2}p_{10}$ for all $n \geq 3$ and use Theorem 3.4.6 in the manuscript

Problem 4 Once again, consider the car insurance of Problem 1, Exercises 3. What is the mean time between two consecutive visits of the customer to the lowest risk category ?

Hint: Use the formula of Lemma 3.4.3 in the manuscript (webpage) to verify that

$$E\left[S_1^{(r)} \left| T_1^{(r-1)} < \infty \right.\right] = E\left[T_1\right].$$

Find $E\left[T_1\right]$ by computing $E\left[T_1 \left| X_1 = i\right.\right], i = 1, 2, 3$ via Theorem 3.2.6 in the manuscript.