Modeling by Stochastic Processes
Exercises 7

Problem 1 Two different Markov chains are defined by the following transition matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(i) Describe the Markov chains by a diagram. Are the Markov chains irreducible ?
(ii) Indentify the communicating classes and classify them as transient or recurrent.

Problem 2 Once more, let S_n be the price of a stock at time n (in weeks) such that

$$S_n = 0\$ \text{ or } 1\$ \text{ or } 2\$, \ n = 0, 1, 2, ...$$

Suppose that the stock prices $(S_n)_{n\geq 0}$ are modelled by a Markov chain with initial distribution $P(S_0 = 1) = 1$ and transition matrix P given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.6 & 0.35 \end{pmatrix}.$$

(i) Has $(S_n)_{n\geq 0}$ an invariant distribution ?
(ii) Which states of the stock price process are periodic or aperiodic

Problem 3 (Success runs) A fair coin is flipped repeatedly, and at time n, we let X_n be the length of the current run of heads. For example, if we get the sequence $HTHHHT$, we have (let $X_0 \equiv 0$)

$$X_1 = 1, \ X_2 = 0, \ X_3 = 1, \ X_4 = 2, \ X_5 = 3, \ X_6 = 0.$$

(i) Describe this sequence of success as a Markov chain and find its invariant distribution.
(ii) Which states are transient or (positive) recurrent ? What are the mean return times of the states ?

Problem 4 (Ehrenfest model of diffusion) Consider two containers containing a total of N gas molecules, connected by a narrow aperture. Each time unit, one of the N molecules is chosen at random to pass through the aperture from one container to the other. Denote by X_n the number of molecules in the first container.

(i) What are the transition probabilities of the Markov chain $X_n, n \geq 0$?
(ii) What is the invariant distribution of the chain ?

Problem 5 Let i and j be communicating states of a Markov chain. Show that i and j must have the same period.