Modeling by Stochastic Processes (STK 2130)
Exercises 8

Problem 1 A Markov chain has the following transition matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}.$$

(i) Describe the Markov chain by a diagram.

(ii) Identify the communicating classes and classify them as periodic or aperiodic, transient or recurrent.

(iii) Calculate the mean return time m_i for each state $i = 1, 2, 3$.

Problem 2 A Markov chain model for the growth and replacement of trees assumes that there are three stages of growth based on the size of the tree: young tree, mature tree, and old tree. When an old tree dies, it is replaced by a young tree with probability 0.3. Order the states numerically, $1, 2$ and 3. State 1 is a young tree, state 2 is a mature tree, and state 3 is an old tree. A Markov chain model for the transitions between each of the states over a period of eight years has the following transition matrix:

$$P = \begin{pmatrix} 0.25 & 0.75 & 0 \\ 0 & 0.5 & 0.5 \\ 0.3 & 0 & 0.7 \end{pmatrix}.$$

Transitions occur over an eight-year period. For example, after a period of eight years the probability that a young tree becomes a mature tree is 0.75 an the probability that it remains a young tree is 0.25.

(i) Show that all states of the the Markov chain denoted by $X_n, n \geq 0$ are aperiodic.

(ii) Find the limiting distribution of X_n, that is

$$\lim_{n \longrightarrow \infty} P(X_n = i)$$

for all $i = 1, 2, 3$

Problem 3 Consider the success run chain of Exerc. 7, Prob. 3. Suppose that the chain has been running for a while and is currently in state 10.

(i) What is the expected number of steps until the chain is back at state 10 ?

(ii) What is the expected number of times the chain visits state 9 before it is back at 10 ?

Hint: See Lemma 3.5.10 in the manuscript.

Problem 4 Once more cast a glance at the stock price process S_n of Exerc. 5, Prob. 2 with

$$S_n = 0\$ \text{ or } 1\$ \text{ or } 2\$ \text{ or } 3\$$$

for all $n = 0, 1, 2, ...$ (in weeks).

Suppose that $(S_n)_{n\geq 0}$ is a Markov chain with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.4 & 0.1 \\ 0.05 & 0.5 & 0.35 & 0.1 \\ 0.01 & 0.6 & 0.3 & 0.09 \end{pmatrix}.$$

Study the long-time behaviour of the stock prices starting in $i = 1, 2$ or 3, i.e. what is

$$P(S_n = i\$ \,|S_0 = i\$)$$

as $n \longrightarrow \infty$ for $i = 1, 2, 3$?