Problem 1

1] Let $\mathbb{P} = \delta_\omega$ for $\omega \in \Omega$

where $\delta_\omega(A) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$

for an event A

Verifies condition i – iii on p. 222 in Norris:

c) We have $\mathbb{P}(\Omega) = \delta_\omega(\Omega)$

$= 1$, as $\omega \in \Omega$

and $\mathbb{P}(\emptyset) = 0$ as $\omega \notin \emptyset$.

ii) Let A_1, A_2 be disjoint events.

We have $\mathbb{P}(A_1 \cup A_2) = \begin{cases} 1, & \omega \in A_1 \cup A_2 \\ 0, & \omega \notin A_1 \cup A_2 \end{cases}$

$= \begin{cases} 2, & \omega \in A \cap B \\ 1, & \omega \in (A \cup B) \setminus (A \cap B) \\ 0, & \text{else} \end{cases}$

[as $A \cap B = \emptyset$]

$= \delta_\omega(A_1) + \delta_\omega(A_2)$.

a) Suppose $A_1 \subseteq A_2 \subseteq \dots$ are events. Either $\omega \in \bigcup_{i=1}^{\infty} A_i$ or not. First consider $\omega \notin \bigcup_{i=1}^{\infty} A_i$

$\Rightarrow \omega \notin A_n$ for all n

$\Rightarrow \mathbb{P}(A_n) = 0$ for all n

$\Rightarrow \mathbb{P}(A_n) \uparrow \mathbb{P}\left(\bigcup_{i=0}^{\infty} A_i\right) = 0$

Converges to and increasing

Now consider $\omega \in \bigcup_{i=1}^{\infty} A_i$

$\Rightarrow$ There exist a N such that $\omega \in A_N$

$\Rightarrow \omega \in A_n$ for all $n \geq N$

(as $A_1 \subseteq A_2 \subseteq \dots$)

$$\Rightarrow \mathbb{P}(A_n) = \begin{cases} 1, & n \geq N \\ 0, & \text{else} \end{cases}$$

$$\uparrow \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = 1$$

$\Rightarrow \mathbb{P} = \delta_\omega$ is a probability measure

Problem 1.2

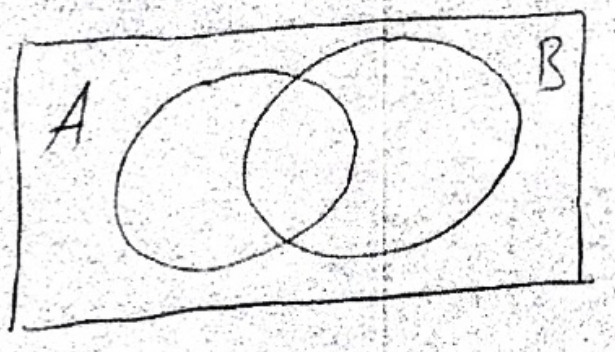
Suppose A, B are events.

Hyp $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$.

Follows from the well known

(*) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$
$- \mathbb{P}(A \cap B)$
$\leq \mathbb{P}(A) + \mathbb{P}(B)$

as $\mathbb{P}(\cdot) \geq 0$. Let's prove this relation. Out of the three axioms for probability measures, clearly only $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ when $E \cap F = \emptyset$ can be usefull in proving this. To use this, we will decompose $A \cup B$ into disjoint parts.



We have $A \cup B = [A \setminus (A \cap B)] \cup [B \setminus (A \cap B)] \cup [A \cap B]$

$= [A \cap B^c] \cup [B \cap A^c] \cup [A \cap B]$ yielding a disjoint decomposition

$\Rightarrow P(A \cup B) = P(A \cap B^c) + P(B \cap A^c) + P(A \cap B)$

Now notice that

$A = [A \cap B] \cup [A \cap B^c]$

$\Rightarrow P(A) = P(A \cap B) + P(A \cap B^c)$

$\Rightarrow P(A \cap B^c) = P(A) - P(A \cap B)$

Symmetrically is $P(B \cap A^c)$

$= P(B) - P(A \cap B)$ proving the $(*)$ relation through substitution.

$A \setminus (A \cap B)$

$= A \cap [A \cap B]^c$

$= A \cap [A^c \cup B^c]$

↑ De Morgan

$= [A \cap A^c] \cup [A \cap B^c]$

$= \emptyset \cup [A \cap B^c]$

$= A \cap B^c$

Problem 2

i) Let $X \sim \text{Poisson}(\lambda)$. That is,

$$\mathbb{P}(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n = 0, 1, \ldots$$

$$\Rightarrow \mathbb{E}X = \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} = \sum_{n=1}^{\infty} \frac{\lambda \lambda^{n-1}}{(n-1)!} e^{-\lambda}$$

$$= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = \lambda \mathbb{P}(X \geq 0) = \lambda$$

ii) Let $X \sim \text{Exp}(\lambda)$. That is,

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0$$

$$\Rightarrow \mathbb{E}X = \int_0^{\infty} x f_X(x) \, dx$$

$$= \int_0^{\infty} x \lambda e^{-\lambda x} \, dx \quad \underline{\text{Partial integration}}$$

$$\left(\frac{\lambda}{-\lambda} e^{-\lambda x} x \right) \Big|_0^{\infty} - \int_0^{\infty} \left(\frac{\lambda}{-\lambda} \right) e^{-\lambda x} \, dx$$

$$= -\lim_{x \to \infty} x e^{-\lambda x} + \int_0^{\infty} e^{-\lambda x} \, dx$$

$$= \frac{-1}{\lambda} \left[e^{-\lambda x} \right]_0^{\infty} = 1/\lambda$$

Problem 3

Let $G \in [0, \infty)$ and $F_T \sim \text{Poisson}(\lambda)$

$$\Rightarrow \mathbb{E} \max(G - F_T, 0)$$

$$= \sum_{n=0}^{\infty} \max(G - n, 0) \, \mathbb{P}(F_T = n)$$

$$= \sum_{n=0}^{G-1} (G - n) \frac{\lambda^n}{n!} e^{-\lambda},$$

which for $G = 4$, $\lambda = 7$ is

$$e^{-7} \left(4 \frac{7^0}{0!} + 3 \frac{7^1}{1!} + 2 \frac{7^2}{2!} + 1 \frac{7^3}{3!} \right)$$

$$\approx 0.1196 \text{ mil. \$}$$