Opgave 1 X_n vore en forgreinings-
proces med Poissonfordeling

$$\mathbb{P}(N=k) = \frac{1}{3}\left(\frac{2}{3}\right)^k, \quad k = 0, 1, \dots$$

a) $\mathbb{E}X = (\mathbb{E}N)^n = \left[\frac{1}{3}\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k\right]^n$

Her $\sum_{n=1}^{\infty} k\left(\frac{2}{3}\right)^k = \frac{2}{3}\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k-1}$

$$= \frac{2}{3}\sum_{k=1}^{\infty}\left(\frac{\partial}{\partial x} x^k\right)\Big|_{x=2/3} = \frac{2}{3}\frac{\partial}{\partial x}\left[\sum_{k=0}^{\infty} x^k\right]\Big|_{x=2/3}$$

$$= \frac{2}{3}\frac{\partial}{\partial x}\left(\frac{1}{1-x}\right)\Big|_{x=2/3} = \frac{2}{3}(-1)\frac{(-1)}{(1-x)^2}\Big|_{x=1/3} = \frac{2/3}{(1/3)^2}$$

$$= 2 \cdot 3 = 6 \Rightarrow \mathbb{E}X = (6/3)^n = 2^n.$$

b) $\mathbb{P}(X_n = 0) = \varphi^{(n)}(0)$ Finn uten denne

$$\varphi(t) = \mathbb{E}\, t^N = \sum_{k=0}^{\infty} t^k \frac{1}{3}\left(\frac{2}{3}\right)^k$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}(2t/3)^k = \frac{1}{3}\frac{1}{1-\frac{2}{3}t} = \frac{1}{3-2t}$$

Her $\varphi(0) = 1/3$, $\varphi^{(2)}(0) = \frac{1}{3-2\varphi(0)}$

$$\frac{1}{3-2/3} = \frac{3}{7}, \quad \varphi^{(3)}(0) = \frac{1}{3-\frac{6}{7}} = \frac{7}{15},$$

$$\varphi^{(4)}(0) = \frac{1}{3-2\frac{7}{15}} = \frac{15}{31}$$

$$\varphi^{(5)}(0) = \frac{1}{3-2\cdot\frac{15}{31}} = \frac{31}{63}. \quad \text{Se at}$$

som $\varphi^{(4)}(0) \approx \frac{1}{2}$,

og at $\varphi^{(n)}(0) = \dfrac{2^n - 1}{2^{n+1} - 1}$.

Viser dette ved induksjon: La $a_n = \varphi^{(n)}(0)$

Har $a_1 = \frac{1}{3} = \dfrac{2^1 - 1}{2^2 - 1}$. Anta

$a_n = \dfrac{2^n - 1}{2^{n+1} - 1}$. Vil da $a_{n+1} = \dfrac{2^{n+1} - 1}{2^{n+2} - 1}$?

Har $a_{n+1} = \dfrac{1}{3 - 2a_n} = \dfrac{1}{3 - 2\,\frac{2^n - 1}{2^{n+1} - 1}}$

$= \dfrac{2^{n+1} - 1}{3(2^{n+1} - 1) - 2(2^n - 1)} = \dfrac{2^{n+1} - 1}{3 \cdot 2^{n+1} - 3 - 2^{n+2} + 2}$

$= \dfrac{2^{n+1} - 1}{2^{n+1}(3 - 1) - 1} = \dfrac{2^{n+1} - 1}{2^{n+2} - 1}$, altså

viser $\varphi^{(n)}(0) = \dfrac{2^n - 1}{2^{n+1} - 1}$.

c) Har $q = \lim_{n \to \infty} \varphi^{(n)}(0) = \lim_{n \to \infty} \dfrac{2^n - 1}{2^{n+1} - 1}$

$= \lim_{n \to \infty} \dfrac{1 - 2^{-n}}{2 - 2^{-n}} = \dfrac{1}{2}$.

Oppgave 2 Dette er en forgreningsprosess B med forbehavet $N = \sum_{i=1}^{Z} B_i$,
$B_i \sim Bin(1, p)$, $Z \sim Poisson(\lambda)$
den Z, B_i er uavhengige, og $\lambda = 7$

a) Vet at $q = \mathbb{P}(X_n = 0 \text{ for en } n)$
$= 1 \Leftrightarrow \mu < 1$ der $\mu = \mathbb{E}N$
$= \mathbb{E}\sum_{i=1}^{Z} B_i = \mathbb{E}[\mathbb{E}[\sum_{i=1}^{Z} B_i \mid Z]]$
$= \mathbb{E}[(Z+1)p] = (\lambda+1)p$
$\Rightarrow \mu < 1 \Leftrightarrow (\lambda+1)p < 1 \Leftrightarrow p < \frac{1}{\lambda+1}$
$= 1/8$

b) Anta $p = 20\%$. Har $\mathbb{P}(X_2 > 0)$
$= \ldots 1 - \mathbb{P}(X_2 = 0)$, der $\mathbb{P}(X_2 = 0)$
$= \varphi^{(1)}(0)$ Har $\varphi(t) = \mathbb{E}t^N$
$= \mathbb{E}t^{\sum_{i=1}^{Z} B_i} = \mathbb{E}\prod_{i=1}^{Z} t^{B_i} =$
$\mathbb{E}[\mathbb{E}[\prod_{i=1}^{Z} t^{B_i} \mid Z]] \overset{\text{uavh.}}{=\!=} \mathbb{E}\prod_{i=1}^{Z} \mathbb{E}t^{B_i}$
Har $\mathbb{E}t^{B_i} = \sum_{n=0}^{1} t^n \mathbb{P}(B_i = n)$
$= \mathbb{P}(B_i = 0) + t\,\mathbb{P}(B_i = 1) = (1-p) + tp$
$\Rightarrow \varphi(t) = \mathbb{E}[(1-p) + (tp)]^Z$

$$= \mathbb{E}\left[1 + \rho(t-1)\right]^z$$

$$= \psi(1 + \rho(t-1))$$

der $\psi(t) = \mathbb{E}\, t^z = \sum_{n=0}^{\infty} t^n \frac{\lambda^n e^{-\lambda}}{n!}$

$$= e^{\lambda(t-1)} \sum_{n=0}^{\infty} \frac{(tn)^n e^{-\lambda t}}{n!} = e^{\lambda(t-1)}$$

$\Rightarrow \varphi(t) = \ldots$

$\exp[\lambda\, \rho(t-1)]$

Sett så inn konkrete tallverdier

i a): finn $\varphi(\varphi(0))$.

b: finn q slik at $q = \varphi(q)$, som krever Newtons metode for å løse.

Eller se Ex. 4.1.3 i manus.

Oppgave 3 La $N \sim \text{Poisson}(\lambda)$

Har $q = \varphi(q)$ for $\varphi(t)$

$$= \mathbb{E}\, t^N = \sum_{n=0}^{\infty} t^n \mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} t^n \frac{\lambda^n e^{-\lambda}}{n!} = e^{\lambda(t-1)} \sum_{n=0}^{\infty} \frac{(t\lambda)^n e^{-\lambda t}}{n!}$$

$$= e^{\lambda(t-1)} \Rightarrow q = e^{\lambda(q-1)}$$

Kan dermed løse $\lambda(q-1) = \log q$ via f.eks Newtons metode for gitt λ.

Oppgave 4 Argumentet på s 72173 er at siden 0 er absorberende, mens alle andre tilstander er i en åpen klasse, gir transienten at prosessen med sannsynlighet 1 etter hvert aldri kommer tilbake til den den startet hvis man ikke starter i 0. Siden dette gjelder tilstand 1, 2 og 3 er den eneste muligheten at

$$\lim_{n \to \infty} S_n = 0.$$