Oppgave 1 La X_n være en forgreiningsprosess med sirkumsfordeling N hvor $\mathbb{E}N = \mu$, $\mathrm{Var}\, N = \sigma^2$.

a) Hyp: $\mathrm{Var}\, X_2 = \begin{cases} 2\sigma^2, & \mu = 1 \\ \dfrac{\sigma^2(\mu^2-1)\mu}{\mu-1}, & \mu \neq 1. \end{cases}$

Har $P_{X_2}(t) \overset{\text{Def}}{=} \mathbb{E}\, t^{X_2} = \varphi(\varphi(t))$

- for $\varphi(t) = \mathbb{E}\, t^N$. Har:

$$\mathrm{Var}\, X_2 = P''_{X_2}(1) + P'_{X_2}(1) - [P'_{X_2}(1)]^2$$

$$= \left[\frac{\partial^2}{\partial t^2} \varphi(\varphi(t)) + \frac{\partial}{\partial t} \varphi(\varphi(t)) - \left[\frac{\partial}{\partial t} \varphi(\varphi(t)) \right]^2 \right]\Bigg|_{t=1}$$

Har:

- $P'_{X_2}(1) = \frac{\partial}{\partial t} \varphi(\varphi(t))\Big|_{t=1} = \varphi'(\varphi(t)) \cdot \varphi'(t)\Big|_{t=1}$

$$= \varphi'(1) \cdot \varphi'(1) = [\varphi'(1)]^2$$

$\varphi(1) = \mathbb{E}\, 1^N = 1$

Har $\varphi'(t)\Big|_{t=1} = \frac{\partial}{\partial t} \sum_{i=0}^{\infty} t^i \mathbb{P}(N=i)\Big|_{t=1}$

$$= \sum_{i=1}^{\infty} i\, t^{i-1} \mathbb{P}(N=i)\Big|_{t=1}$$

$$= \sum_{i=1}^{\infty} i\, \mathbb{P}(N=i) = \sum_{i=0}^{\infty} i \mathbb{P}(N=i) = \mu$$

$\Rightarrow P'_{X_2}(1) = \mu^2.$

Har videre $\frac{\partial^2}{\partial t^2} \varphi(\varphi(t))\Big|_{t=1}$

$= \left[\frac{\partial}{\partial t} \varphi'(\varphi(t)) \varphi'(t)\right]\Big|_{t=1}$

$= \left[\varphi''(\varphi(t)) \varphi'(t)^2 + \varphi''(t) \varphi'(\varphi(t))\right]\Big|_{t=1}$

$= \varphi''(1) \mu^2 + \varphi''(1) \mu$

$= \varphi''(1)(\mu^2 + \mu) = \varphi''(1)(\mu+1)\mu$

Har $\mathrm{Var}\, N = \varphi''(1) + \varphi'(1) - [\varphi'(1)]^2$

$\Rightarrow \varphi''(1) = \sigma^2 - \varphi'(1) + [\varphi'(1)]^2$

$= \sigma^2 - \mu + \mu^2.$

$\Rightarrow P''_{X_2}(1) = [\sigma^2 - \mu + \mu^2](\mu+1)\mu$

$\Rightarrow \mathrm{Var}\, X_2 = [\sigma^2 - \mu + \mu^2](\mu+1)\mu$

$+ \mu^2 - \mu^4 = \sigma^2(\mu+1)\mu$

$+ \mu^2(\mu-1)(\mu+1) + \mu^2(1-\mu^2)$

$= \sigma^2(\mu+1)\mu + \mu^2(\mu^2-1)$

$- \mu^2(\mu^2-1) = \sigma^2(\mu+1)\mu.$

$= \begin{cases} 2\sigma^2, & \mu = 1 \\ \dfrac{\sigma^2(\mu-1)(\mu+1)\mu}{\mu-1}, & \mu \neq 1 \end{cases}$

b) La $N = \sum_{i=1}^{Z} B_i$, $Z \sim \text{Poisson}(\lambda)$, $\lambda = 10$ /3

$B_i \sim \text{Bin}(1, 20\%)$ med Z, B_i uavh.

Har $\varphi(t) = \mathbb{E}\, t^N = e^{2(t-1)}$

Vet $\mu = \varphi'(1) = 2e^{2(1-1)} = 2$

og $\sigma^2 = \varphi''(1) + \varphi'(1) - (\varphi'(1))^2$.

Har $\varphi''(1) = 4e^{2(1-1)} = 4$

$\Rightarrow \sigma^2 = 4 + 2 - 4 = 2$.

$\Rightarrow \text{Var}\, X_2 = \sigma^2(\mu+1)\mu = 2 \cdot 3 \cdot 2 = 12$.

Oppgave 2 La D_n være etterspørsel i tid n og la

$$X_{n+1} = \begin{cases} (\zeta - D_{n+1})^+, & X_n \leq m \\ (X_n - D_{n+1})^+, & m < X_n \leq \zeta \end{cases}$$

for $\zeta = 2$ og $m \in \{0, 1, \dots\}$.

Antar $D_n \sim$ Poisson(2) i.i.d

a) Finner P for å finne π:

Har $X_n \in \{0, 1, 2\}$, så P er en (3×3) – matrise

Har $p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i)$

$$= \begin{cases} \mathbb{P}([\zeta - D_{n+1}]^+ = j), & i \leq m \\ \mathbb{P}([i - D_{n+1}]^+ = j), & m < i \leq \zeta \end{cases}$$

Er dermed interessert i punktfordelingen

$$p_j(k) = \mathbb{P}([k - D]^+ = j)$$

for $0 \leq k, j \leq 2$ og $D \sim$ Poisson(2)

Forel: Har $\rho_j(k) = 0$ når $k < j$, siden $D \geq 0$.

Anta dermed $k \geq j$, og anta $j \neq 0$:

Har $\mathbb{P}([k-D]^+ = j)$

$= \mathbb{P}(k - D = j) = \mathbb{P}(D = k-j)$

$= \frac{2^{k-j} e^{-2}}{(k-j)!}$ Dessuten er

$\mathbb{P}([k-D]^+ = 0) = \mathbb{P}(D \geq k)$

$= \sum_{l=k}^{\infty} \frac{2^l e^{-2}}{l!} = \left(\sum_{l=0}^{\infty} \frac{2^l e^{-2}}{l!}\right)$

$- \sum_{l=0}^{k-1} 2^l \frac{e^{-2}}{l!} = 1 - \sum_{l=0}^{k-1} \frac{2^l e^{-2}}{l!}$.

Finner deretter P og π i R.

La $\pi(m)$ være ekvilibriumfordelingen hvil P gitt m-verdi.

Har $\pi(1) \approx (59\%, 27\%, 13\%)$,

mens $\pi(0) \approx (66\%, 24\%, 10\%)$

Trenger deretter

$$u_i(m) = \begin{cases} \mathbb{E}((D-i)^+), & i \leq m \\ \mathbb{E}((D-i)^+), & m < i \leq C. \end{cases}$$

Hér $\mathbb{E} \max(D-k, 0)$

$$= \sum_{l=0}^{\infty} \max(l-k, 0) \frac{2^l e^{-2}}{l!}$$

$$= \sum_{l=k}^{\infty} (l-k) \frac{2^l e^{-2}}{l!}$$

$$= \sum_{l=k}^{\infty} l \frac{2^l e^{-2}}{l!} - k \sum_{l=k}^{\infty} \frac{2^l e^{-2}}{l!}$$

$$= 2^l \sum_{l=k}^{\infty} \frac{2^{l-1} e^{-2}}{(l-1)!} - k\left(1 - \sum_{l=0}^{k-1} \frac{2^l e^{-2}}{l!}\right)$$

$$= 2\left(1 - \sum_{l=0}^{k-1} \frac{2^l e^{-2}}{l!} + \frac{2^{k-1} e^{-2}}{(k-1)!}\right.$$

$$\left. - k\left(1 - \sum_{l=0}^{k-1} \frac{2^l e^{-2}}{l!}\right)\right.$$

$$= 2 + \frac{2^k e^{-2}}{(k-1)!} - k - (2-k) \sum_{l=0}^{k-1} \frac{2^l e^{-2}}{l!}$$

$$\Rightarrow \mathbb{E}((D-\zeta)^+) = \mathbb{E}((D-2)^+) = 2e^{-2}.$$

og $\mathbb{E}((D-1)^+) = 1 + 2e^{-2}$

$$- e^{-2} = 1 + e^{-2}$$

$$\Rightarrow \quad u_i(0) = \begin{cases} 2e^{-2}, & i = 0 \\ 1 + e^{-2}, & i = 1, 2 \end{cases}$$

$$\text{og} \quad u_i(1) = \begin{cases} 2e^{-2}, & i = 0, 1 \\ 1 + e^{-2}, & i = 2. \end{cases}$$

Finner $\bar{g}(0) = 0.5659$, $\bar{f}(0) = 0.6585$,

$\bar{g}(1) = 0.3877$, $\bar{f}(1) = 0.8969$.

Her $a = 1$, og $s_m(b) = a\bar{g}(m) + b\bar{f}(m)$ vare gj.snittlig langtidstap. Har $s_m(b)$

$$= 0.5659 + b\bar{f}(m)$$

$$= \begin{cases} 0.5659 + b \cdot 0.6585, & m = 0 \\ 0.3877 + b \cdot 0.8969, & m = 1. \end{cases}$$

Har $s_0(b) < s_1(b) \Leftrightarrow 0 > s_0(b) - s_1(b) = 0.1783 + b \cdot 0.2384$

$$\Leftrightarrow b < \frac{-0.1783}{0.2384} < 0$$

som ikke er mulig $\Rightarrow m = 1$ er best.

b) For vilkårlig $a, b > 0$ er

$$g_m(a,b) = \begin{cases} a\ 0.5659 + b\,0.6585, & m=0 \\ a \cdot 0.3877 + b\ 0.8969, & m=1. \end{cases}$$

Her $g_0(a,b) < g_1(a,b)$

$\Leftrightarrow \quad 0 > 0.1783\,a + b \cdot 0.8969$

som ikke er mulig siden $a, b > 0$

$\Rightarrow m = 1$ er altid best

Oppgave 3

$$\mathbb{P}(N_{t_{n+1}} = i_{n+1} \mid N_{t_n} = i_n)$$

Def $\mathbb{P}(N_{t_{n+1}} = i_{n+1}, N_{t_n} = i_n) / \mathbb{P}(N_{t_n} = i_n)$

Har $\mathbb{P}(N_{t_{n+1}} = i_{n+1}, N_{t_n} = i_n)$

$= \mathbb{P}(N_{t_{n+1}} - N_{t_n} = i_{n+1} - i_n, N_{t_n} - N_0$

$= i_n - 0) = \mathbb{P}(N_{t_{n+1}} - N_{t_n} = i_{n+1} - i_n)$

$\mathbb{P}(N_{t_n} - N_0 = i_n) = \mathbb{P}(N_{t_{n+1}} - N_{t_n}$

$= i_{n+1} - i_n) \mathbb{P}(N_{t_n} = i_n)$

$\Rightarrow \mathbb{P}(N_{t_{n+1}} = i_{n+1} \mid N_{t_n} = i_n)$

$= \mathbb{P}(N_{t_{n+1}} - N_{t_n} = i_{n+1} - i_n)$.

$= \mathbb{P}(N_{t_{n+1}} - N_{t_n} = i_{n+1} - i_n) \prod_{k=-1}^{n} \frac{\mathbb{P}(N_{t_k} - N_{t_{k-1}} = i_k - i_{k-1})}{\mathbb{P}(N_{t_n} - N_{t_{n-1}} = i_k - i_{k-1})}$

nevn $\mathbb{P}\left(\bigcap_{k=-1}^{n+1} \{N_{t_k} - N_{t_{k-1}} = i_{k+1} - i_k\}\right)$

$t_{-1} = 0$, $i_{-1} = 0$, $N_0 = 0$

$/ \mathbb{P}\left(\bigcap_{k=-1}^{n} \{N_{t_k} - N_{t_{n-1}} = i_{k+1} - i_k\}\right)$

$= \mathbb{P}\left(N_{t_{n+1}} = i_{n+1} \cap \left[\bigcap_{k=-1}^{n} \{N_{t_n} - N_{t_{n-1}}\right.\right.$

$= i_{n+1} - i_n \} \; / \; \mathbb{P}\Big(\bigcap_{k=-1}^{n} \{ N_{t_k} - N_{t_{k-1}} \quad \underline{\underline{\text{L10}}}$

$= i_k - i_{k-1} \}$)

Def / let same $\mathbb{P}(N_{t_{n+1}} = i_{n+1} \mid \bigcap_{k=-1}^{n} \{ N_{t_k} - N_{t_{k-1}}$

$= i_k - i_{k-1} \}) = \mathbb{P}(N_{t_{n+1}} = i_{n+1}$

$\mid N_{t_0} = i_0, N_{t_1} = i_1, \dots, N_{t_n} = i_n)$.