# Modeling by Stochastic Processes (STK 2130)
# Solutions to Exercises 12

**Problem 4 (Memoryless property)** Let $T$ be an exponentially distributed random variable with parameter $\lambda > 0$.

Then, using the definition of conditional probabilities gives

$$\begin{aligned} P(T > t+s \,|T > s) &= \frac{P(T > t+s \text{ and } T > s)}{P(T > s)} \\ &= \frac{P(T > t+s\ )}{P(T > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(T > t) \end{aligned}$$

for all $s, t \geq 0$.

**Problem 5** By Remark 5.2.6 in the manuscript we have to solve Kolmogorov's forward equation

$$P'(t) = P(t) \cdot Q,\ P(0) = (\delta_{ij})_{i,j \in I}\,,$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}\ .$$

The latter equation is equivalent to the following system of 16 linear differential equations

$$\frac{d}{dt} p_{ij}(t) = \sum_{k=1}^{4} p_{ik}(t) q_{kj},\ p_{ij}(0) = \delta_{ij}$$

for all $i, j \in I = \{1, 2, 3, 4\}$, where $p_{ij}(t)$ and $q_{ij}$ are the entries of $P(t)$ and $Q$, respectively. Rather than applying systematic theory let us make a guess to find the solution to these equations. Actually, one checks by differentiation w.r.t. time $t$ that

$$\begin{aligned} P(t) &= (p_{ij}(t))_{1 \leq i,j \leq 4} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 - e^{-t} & e^{-t} & 0 & 0 \\ 1 - 2e^{-t} + e^{-2t} & 2e^{-t} - 2e^{-2t} & e^{-2t} & 0 \\ 1 - 3e^{-t} + 3e^{-2t} - e^{-3t} & 3e^{-t} - 6e^{-2t} + 3e^{-3t} & 3e^{-2t} - 3e^{-3t} & e^{-3t} \end{pmatrix} \end{aligned}$$

is the solution.

**Problem 6 (Cramér-Lundberg model)** (i) Calculate the maximum likelihood estimator $\widehat{\lambda}$ based on the Danish fire insurance data from 02/09/1987 until 02/22/1987, that is maximize the likelihood function

$$L(\lambda) = \prod_{j=1}^{n} f_{S_i}(\lambda; S_j)$$

w.r.t. $\lambda > 0$ for *i.i.d.* inter-arrival times (holding times) $S_i \sim Exp(\lambda)$.

So we get

$$L(\lambda) = \prod_{j=1}^{n} \lambda \exp(-\lambda S_j) = \lambda^n \exp(\lambda(S_1 + ... + S_n)) = \lambda^n \exp(\lambda T_n).$$

So $L(\lambda)$ attains the maximum at $\widehat{\lambda} = \frac{n}{T_n}$. Using the data we see that the sample size is given by $n = 10$ and that $T_n = 13$ (days). Hence $\widehat{\lambda} = \frac{10}{13} \approx 0.769$.

(ii) In order to compute the mean and the variance of the arrival times $T_n$, we have to determine the distribution their distribution.

Since $T_n = S_1 + ... + S_n$ with *i.i.d.* $S_i \sim Exp(\widehat{\lambda})$ we obtain

$$P(T_n \leq x) = 1 - e^{-\widehat{\lambda}x} \sum_{k=0}^{n-1} \frac{(\widehat{\lambda}x)^k}{k!}, x \geq 0. \tag{1}$$

Without loss of generality let us verify this for $n = 2$ :

$$P(T_2 \leq x) = P(S_1 + S_2 \leq x) = E[1_{\{0 \leq s_1 + s_2 \leq x\}}(S_1, S_2)] = E[F(S_1, S_2)],$$

where $F$ is the function

$$F(s_1, s_2) = 1_{\{0 \leq s_1 + s_2 \leq x\}}(s_1, s_2) = \begin{cases} 1 & \text{if } 0 \leq s_1 + s_2 \leq x \\ 0 & \text{else} \end{cases}.$$

Since the holding times are *i.i.d.* with exponential joint distribution we can compute the joint density $g(s_1, s_2)$ of $(S_1, S_2)$ :

$$g(s_1, s_2) = f_{S_1}(\widehat{\lambda}; s_1) \cdot f_{S_2}(\widehat{\lambda}; s_2) = \widehat{\lambda} e^{-\widehat{\lambda}s_1} \cdot \widehat{\lambda} e^{-\widehat{\lambda}s_2}.$$

So we can rewrite $E[F(S_1, S_2)]$ as a double integral w.r.t. to the joint density $g(s_1, s_2)$ and get

$$\begin{aligned} P(T_2 \leq x) &= \int_0^\infty \int_0^\infty F(s_1, s_2) g(s_1, s_2) ds_1 ds_2 \\ &= \int_0^x e^{-\widehat{\lambda}s_2} \int_0^{x - s_2} \widehat{\lambda}^2 e^{-\widehat{\lambda}s_1} ds_1 ds_2 \\ &= 1 - e^{-\widehat{\lambda}x} (\sum_{k=0}^{1} \frac{(\widehat{\lambda}x)^k}{k!}). \end{aligned}$$

So we get (1). Differentiation of (1) w.r.t. $x$ on both sides finally shows that the density $f_{T_n}$ of $T_n$ is given by the density of the Gamma function with shape parameter $\alpha = n$ and scale parameter $\lambda = \widehat{\lambda}$, that is

$$f_{T_n}(x) = e^{-\widehat{\lambda}x} \widehat{\lambda}^n \frac{x^{n-1}}{(n-1)!}, x > 0.$$

Using this we see that

$$E[T_n] = \frac{\alpha}{\lambda} = \frac{n}{\widehat{\lambda}} = 1.3 \cdot n$$

and

$$Var[T_n] = \frac{\alpha}{\lambda^2} = \frac{n}{\widehat{\lambda}^2} = (1.3)^2 \cdot n.$$

As a special case (i.e. $n = 1$) we get for the expected inter-arrival time of the occurence of fire losses

$$E[S_1] = 1.3 \text{ days.}$$