Problem 1

a) Let X_n denote the weather at day n and let 0 be "sunny" and 1 be "rainy". We then have

$$\mathbb{P}(X_{n+1} = 0 \mid X_n = 0) = 0.9$$

which implies that

$$\mathbb{P}(X_{n+1} = 1 \mid X_n = 0) = 0.1$$

as $1 = \mathbb{P}(X_{n+1} \in \{0,1\} \mid X_n = 0)$

$$= \mathbb{P}(\{X_{n+1} = 0\} \cup \{X_{n+1} = 1\} \mid X_n = 0)$$

$$= \frac{\mathbb{P}([\{X_{n+1}=0\} \cap \{X_n=0\}] \cup [\{X_{n+1}=1\} \cap \{X_n=0\}])}{\mathbb{P}(X_n = 0)},$$

$$= \frac{\mathbb{P}(\{X_{n+1}=0\} \cap \{X_n=0\})}{\mathbb{P}(X_n=0)} + \frac{\mathbb{P}(\{X_{n+1}=1\} \cap \{X_n=0\})}{\mathbb{P}(X_n=0)}$$

$$= \mathbb{P}(X_{n+1}=0 \mid X_n=0) + \mathbb{P}(X_{n+1}=1 \mid X_n=0)$$

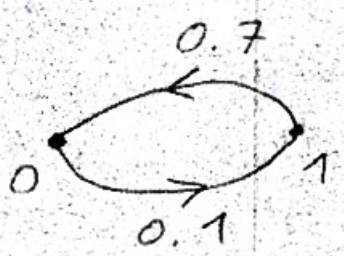
$$\Rightarrow \mathbb{P}(X_{n+1}=1 \mid X_n=0) = 1 - \mathbb{P}(X_{n+1}=0 \mid X_n=0) = 0.1$$

$\{X_{n+1}=0\} \cap \{X_{n+1}=1\} = \emptyset$!

Further, $\mathbb{P}(X_{n+1} = 1 \mid X_n = 1) = 0.3$

$\Rightarrow \mathbb{P}(X_{n+1} = 0 \mid X_n = 1) = 0.7$ [2]

The actual Markov property is in fact stated implicitly in the description of the problem.



ii) The actual names of the days in question is only relevant when counting their distance. We want $\mathbb{P}(X_{n+2} = 0 \mid X_n = 0)$, but only know the transition probabilities from one day to the next. This is solved by the following technique:

$$\mathbb{P}(X_{n+2}=0 \mid X_n=0)$$

$$= \mathbb{P}(\{X_{n+2}=0\} \cap [\{X_{n+1}=0\} \cup \{X_{n+1}=1\}] \mid X_n=0)$$

Distributive law of set theory

$$= \mathbb{P}([\{X_{n+2}=0\} \cap \{X_{n+1}=0\}] \cup [\{X_{n+2}=0\} \cap \{X_{n+1}=1\}] \mid X_n=0)$$

As $\Omega = \{X_{n+1}=0\} \cup \{X_{n+1}=1\}$

$$= \mathbb{P}(\{X_{n+2}=0\} \cap \{X_{n+1}=0\} \mid X_n=0) + \mathbb{P}(\{X_{n+2}=0\} \cap \{X_{n+1}=1\} \mid X_n=0)$$

$\{X_{n+2}=0\} \cap \{X_{n+1}=0\}$ is disjoint from $\{X_{n+2}=0\} \cap \{X_{n+1}=1\}$

$$= \mathbb{P}(X_{n+2}=0 \mid \{X_{n+1}=0\} \cap \{X_n=0\}) \cdot \mathbb{P}(X_{n+1}=0 \mid X_n=0) + \mathbb{P}(X_{n+2}=0 \mid \{X_{n+1}=1\} \cap \{X_n=0\}) \cdot \mathbb{P}(X_{n+1}=1 \mid X_n=0)$$

- This is proved on a following page!

Markov property

$$\mathbb{P}(X_{n+2}=0 \mid X_{n+1}=0)$$
$$\mathbb{P}(X_{n+1}=0 \mid X_n=0)$$
$$+ \mathbb{P}(X_{n+2}=0 \mid X_{n+1}=1)$$
$$\mathbb{P}(X_{n+1}=1 \mid X_n=0)$$
$$= 0.9 \cdot 0.9 + 0.7 \cdot 0.1 = 88\%.$$

This calculation will be much simpler once we learn more about Markov-Chains!

iii) Again, the actual naming of the days are irrelevant. We want

$$\mathbb{P}(\{X_{n+2}=0\} \cap \{X_{n+1}=0\} \mid X_n=0) = \mathbb{P}(X_{n+2}=0 \mid \{X_{n+1}=0\} \cap \{X_n=0\})\, \mathbb{P}(X_{n+1}=0 \mid X_n=0)$$
$$\overset{\text{Markov prop.}}{=} \mathbb{P}(X_{n+2}=0 \mid X_{n+1}=0) \cdot \mathbb{P}(X_{n+1}=0 \mid X_n=0)$$
$$= 0.9^2 = 81\%.$$

Proof is given on the next page!

In problem 1 ii, the following [5] was used:

$$\mathbb{P}(A \cap B \mid C)$$

$$= \mathbb{P}(A \mid B \cap C)\,\mathbb{P}(B \mid C)$$

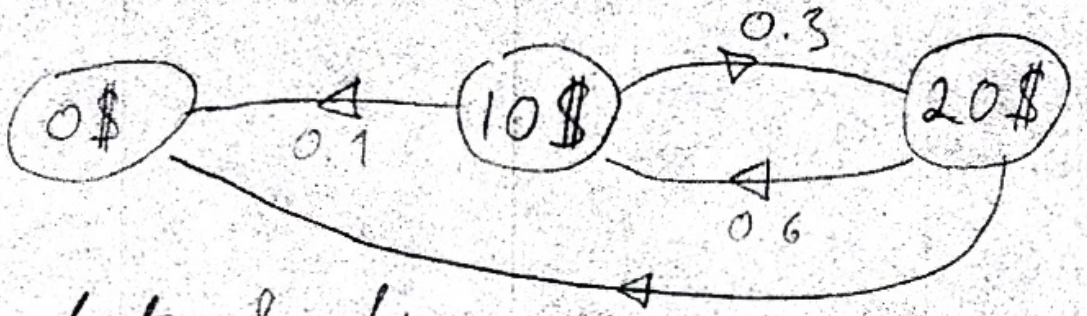
Proof: $\mathbb{P}(A \mid B \cap C)\,\mathbb{P}(B \mid C)$

$$\overset{\text{Def}}{=} \frac{\mathbb{P}(A \cap B \cap C)}{\cancel{\mathbb{P}(B \cap C)}} \cdot \frac{\cancel{\mathbb{P}(B \cap C)}}{\mathbb{P}(C)}$$

$$= \frac{\mathbb{P}([A \cap B] \cap C)}{\mathbb{P}(C)} \overset{\text{Def}}{=} \mathbb{P}(A \cap B \mid C)$$

This is very useful!

Problem 2



Want to find:

$$\mathbb{P}(\{S_1 = 20\} \cup \{S_2 = 20\} \cup \{S_3 = 20\} \mid S_0 = 10).$$

As $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$

from the inclusion/exclusion principle (Check out Wikipedia!) we could calculate this probability through brute force; but this chain has an especially simple structure

which allows us to avoid doing this:

Notice that

$$\mathbb{P}(S_1 = 20 \text{ or } S_2 = 20 \text{ or } S_3 = 20 \mid S_0 = 10) = \mathbb{P}(\{S_1 = 20\} \text{ or } \{S_1 = 10, S_2 = 20\} \text{ or } \{S_1 = 10, S_2 = 10, S_3 = 20\} \mid S_0 = 10)$$

(as all other paths to the 20 $-state has zero probability!)

These events are disjoint!

$$\mathbb{P}(S_1 = 20 \mid S_0 = 10) + \mathbb{P}(\{S_1 = 10\} \cap \{S_2 = 20\} \mid S_0 = 10) + \mathbb{P}(\{S_1 = 10\} \cap \{S_2 = 10\} \cap \{S_3 = 20\} \mid S_0 = 10)$$

See p. 5!! $0.3 + \mathbb{P}(S_2 = 20 | \{S_1 = 10\} \cap \{S_0 = 10\}) \cdot \mathbb{P}(S_1 = 10 | S_0 = 10)$

$+ \mathbb{P}(\{S_3 = 20\} \cap \{S_2 = 10\} | \{S_1 = 10\} \cap \{S_0 = 10\})$

$\cdot \mathbb{P}(S_1 = 10 | S_0 = 10)$

Markov property $0.3 + 0.3 \cdot 0.6 +$

$0.6 \cdot \mathbb{P}(\{S_3 = 20\} \cap \{S_2 = 10\} | S_1 = 10)$

$= 0.3 \cdot 1,6 + 0.6 \cdot \mathbb{P}(S_3 = 20 | \{S_2 = 10\} \cap \{S_1 = 10\}) \cdot \mathbb{P}(S_2 = 10 | S_1 = 10$

Markov property $0.48 + 0.6^2 \cdot \mathbb{P}(S_3 = 20 | S_2 = 10)$

$= 0.48 + 0.6^2 \cdot 0.3 = 58.8\%$

Problem 3

Let $(X_n)_{n\geq 0}$ be i.i.d. living on a countable index-set I with probability 1.

Hyp: $(X_n)_{n\geq 1}$ is a Markov chain

Denote $\mathbb{P}(X_n = i)$ by λ_i for $i \in I$. This can be defined without letting λ_i depend on n as $\mathbb{P}(X_n = i) = \mathbb{P}(X_m = i)$ from the i.i.d. assumption.

The first demand for accepting $(X_n)_{n\geq 1}$ (see Def. 3.1.2 in the man) as Markov is trivial. For the second, the independence assumption yields

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \dots, X_n = i_n)$$

$$= \mathbb{P}(X_{n+1} = i_{n+1}) = \lambda_{i_{n+1}}$$

which clearly is a distribution in i_{n+1}!

Problem 4

Let $(X_n)_{n\geq 0}$ be i.i.d with distribution λ_i, $i \in I$ where I is countable. Then let

$$S_n = X_0 + X_1 + \cdots + X_n = \sum_{k=0}^{n} X_i$$

for $n \geq 0$. **Is $(S_n)_{n\geq 0}$ Markov?**

We have $\mathbb{P}(S_{n+1} = i_{n+1} \mid S_0 = i_0, \ldots, S_n = i_n)$

$$= \frac{\mathbb{P}(S_{n+1} = i_{n+1}, S_n = i_n, \ldots, S_0 = i_0)}{\mathbb{P}(S_n = i_n, \ldots, S_0 = i_0)}.$$

Now notice that

$\mathbb{P}(S_n = i_n, S_{n-1} = i_{n-1}, \ldots, S_0 = i_0)$

$= \mathbb{P}(X_0 = i_0, X_1 = i_1 - i_0, \ldots, X_n = i_n - i_{n-1})$ as

$$S_n - S_{n-1} = \sum_{k=0}^{n} X_k - \sum_{k=0}^{n-1} X_k$$

$$= X_n .$$

As $(X_n)_{n\geq 0}$ is assumed to be independent, we get:

$$\frac{\mathbb{P}(S_{n+1} = i_{n+1}, S_n = i_n, \dots, S_0 = i_0)}{\mathbb{P}(S_n = i_n, \dots, S_0)}$$

$$= \frac{\mathbb{P}(X_0 = i_0, X_1 = i_1 - i_0, \dots, X_n = i_n - i_{n-1})}{\mathbb{P}(X_0 = i_0, X_1 = i_1 - i_0, \dots, X_{n-1} = i_{n-1} - i_{n-2})}$$

$$= \mathbb{P}(X_n = i_n - i_{n-1} \mid X_{n-1} = i_{n-1} - i_{n-2}, \dots, X_0 = i_0) = \mathbb{P}(X_n = i_n - i_{n-1})$$

$$= \lambda_{i_n - i_{n-1}},$$ which is always a distribution in i_n as i_n is allowed to live

on $\mathbb{Z}$ (if $I \subseteq \mathbb{Z}$, or the algebraic closure of I)

$\Longrightarrow$ Yes, $(S_n)_{n \geq 0}$ is a Markov Chain.

Problem 5

Let $B_1, B_2, \ldots$ be disjoint events with $\bigcup_{n=1}^{\infty} B_n = \Omega$. Let A be an event

Assume $\mathbb{P}(A \mid B_n) = p$ for all n.

Hyp: $\mathbb{P}(A) = p$.

We have $\mathbb{P}(A) = \mathbb{P}(A \cap \Omega)$

$= \mathbb{P}\left(A \cap \left[\bigcup_{n=1}^{\infty} B_n\right]\right)$

Distributive law of set theory: $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A \cap B_n\right)$

$= \sum_{n=1}^{\infty} \mathbb{P}(A \cap B_n)$

$\{B_n\}_{n\geq 1}$ disjoint implies $\{B_n \cap A\}_{n\geq 1}$ disjoint!

Def of cond. prob: $\sum_{n=1}^{\infty} \mathbb{P}(A \mid B_n)\,\mathbb{P}(B_n)$

$= \sum_{n=1}^{\infty} p\,\mathbb{P}(B_n)$

$\{B_n\}_{n\geq 1}$ disjoint: $p\,\mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = p \cdot 1 = p$

Now, assume i: X and Y are independent.

Then $\mathbb{P}(X=i \mid Y=j) = \mathbb{P}(X=i)$ per definition of independence, which is clearly independent of j.

Now, assume ii: $\mathbb{P}(X=i \mid Y=j) = p_i$.

Notice that $B_n = \{\omega \in \Omega : Y(\omega) = j\} = \{Y=j\}$ are disjoint, as Y can only has one value per realization and that $\bigcup_{n=1}^{\infty} B_n = \Omega$, as Y has some value for each $\omega \in \Omega$. Thus, $\Omega \subseteq \bigcup_{n=1}^{\infty} B_n$ as $\omega \in \Omega$

$\Rightarrow$ There exist an i such that $Y(\omega) = i \Rightarrow \omega \in B_i$.

From the previous statement, it follows that $\mathbb{P}(X=i) = \mathbb{P}(X=i \mid Y=j) = p_i$. Thus X and Y are independent.