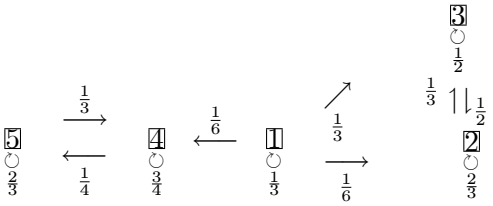
Modeling by Stochastic Processes (STK 2130)
Løsningsforslag

Problem 1

(i)



(ii) As one observes from the diagram the communicating classes are given by $\{1\}, \{2,3\}$ and $\{4,5\}$. Here $\{2,3\}$ and $\{4,5\}$ are "traps", i.e. closed classes.

(iii)

$$P(X_3 = j \,|X_0 = i) \overset{\text{Th. 3.1.5 in the manuscript}}{=} p_{ij}^{(3)},$$

where

$$\begin{aligned} P^3 &= PPP \overset{def}{=} (p_{ij}^{(3)})_{1\le i,j\le 5} \\ &\overset{\text{matrix mult.}}{=} \begin{pmatrix} 1/27 & 11/27 & 17/54 & 145/864 & 7/96 \\ 0 & 65/108 & 43/108 & 0 & 0 \\ 0 & 43/72 & 29/72 & 0 & 0 \\ 0 & 0 & 0 & 347/576 & 229/576 \\ 0 & 0 & 0 & 229/432 & 203/432 \end{pmatrix}. \end{aligned}$$

Problem 2

(i) We want to find $h_i^0 \overset{def}{=} P_i(\text{hit } 0)$ for all $i = 1, 2, 3$. Th. 3.2.4 in the manuscript (see webside) shows that h_i^0, $i \in I = \{0, 1, 2, 3\}$ is the minimal non-negative solution of the linear equation

$$\begin{cases} h_i^0 = 1, & \text{if } i = 0 \\ h_i^0 = \sum_{j=0}^{3} p_{ij} h_j^0, & \text{if } i \neq 0 \end{cases}.$$

Using the transition probabilities we get

$$\begin{cases} h_1^0 = 0.04 + 0.41h_1^0 + 0.35h_2^0 + 0.2h_3^0 \\ h_2^0 = 0.03 + 0.35h_1^0 + 0.37h_2^0 + 0.25h_3^0 \\ h_3^0 = 0.03 + 0.3h_1^0 + 0.4h_2^0 + 0.27h_3^0 \end{cases}.$$

Substitution of variables yields the unique (and therefore minimal) solution

$$h_1^0 = h_2^0 = h_3^0 = 1.$$

(ii) By Th. 3.2.6 in the manuscript (see webside) the mean hitting times $k_i^0, i = 0, ..., 3$ are the minimal non-negative solution of

$$\begin{cases} k_i^0 = 0, & \text{if } i = 0 \\ k_i^0 = 1 + \sum_{j=1}^{3} p_{ij} k_j^0, & \text{if } i \neq 0 \end{cases}.$$

This implies that

$$\begin{cases} k_1^0 = 1 + 0.41k_1^0 + 0.35k_2^0 + 0.2k_3^0 \\ k_2^0 = 1 + 0.35k_1^0 + 0.37k_2^0 + 0.25k_3^0 \\ k_3^0 = 1 + 0.3k_1^0 + 0.4k_2^0 + 0.27k_3^0 \end{cases}.$$

So using substitution gives the unique solution

$$k_1^0 = 29.447, k_2^0 = 29.763, k_3^0 = 29.779 \text{ in weeks.}$$

(iii) The mean hitting times $k_i^A, i = 0, ..., 3$ are the minimal (possibly infinite) solution of

$$\begin{cases} k_i^A = 0, & \text{if } i \in A = \{3\} \\ k_i^A = 1 + \sum_{j=0,1,2} p_{ij} k_j^A, & \text{if } i \in \{0, 1, 2\} \end{cases}.$$

The latter equation becomes

$$\begin{cases} k_i^A = 0 \text{ for } i = 3 \\ k_0^A = 1 + k_0^A \ (\Longrightarrow k_0^A = \infty) \\ k_1^A = 1 + 0.04k_0^A + 0.41k_1^A + 0.35k_2^A \\ k_2^A = 1 + 0.03k_0^A + 0.33k_1^A + 0.4k_2^A \end{cases}.$$

So it follows $k_1^A = \infty$.

On the other hand, since there is a strictly positive probability which leads to zero, one could also argue that

$$k_1^A \overset{def}{=} E[H^A \,|S_0 = 1] \geq E[H^A 1_{\{S_1=0\}} \,|S_0 = 1] = \infty.$$

(iv) $\{0\}$ is a finite closed class. Therefore the state 0 must be recurrent (Rem. 3.4.6. (ii) in the manuscript). On the other hand, every recurrent class is closed (Rem. 3.4.6 (i)). Since the class $\{1, 2, 3\}$ is not closed it cannot be recurrent. Thus the states $1, 2, 3$ are transient.

Problem 3 We have that

$$P = \begin{pmatrix} 0.1 & 0.7 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0.7 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0.7 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.7 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0.7 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0.7 & 0.2 \\ 0.2 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0.7 \\ 0.7 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0.1 \end{pmatrix}.$$

On the other hand, we we get

$$P(i_4 = j \,|i_0 = i) \overset{\text{Th. 3.1.5 in the manuscript}}{=} p_{ij}^{(4)}.$$

So

$$P(i_4 = 2 \,|i_0 = 0) = 3.02\%.$$

Problem 4 (i) The state space can be chosen to be $I = \{0, 3, 6, 9, 12, 15\}$.Then the Markov chain $X_n \in I$, which discribes the wealth of the gambler on play n, has the following transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We are interested to find the hitting probability h_3^{15} by applying Th. 3.2.4 in the manuscript. The latter boils down to solving the following system of linear equations:

$$\begin{cases} h_{20}^{20} = 1 \\ h_0^{20} = h_0^{20} + 0h_4^{20} + 0h_8^{20} + 0h_{12}^{20} + 0h_{16}^{20} + 0h_{20}^{20} \\ h_4^{20} = 0.5h_0^{20} + 0h_4^{20} + 0.5h_8^{20} + 0h_{12}^{20} + 0h_{16}^{20} + 0h_{20}^{20} \\ h_8^{20} = 0.5h_0^{20} + 0.5h_{16}^{20} \\ h_{12}^{20} = 0.5h_4^{20} + 0.5h_{20}^{20} \\ h_{16}^{20} = 0.5h_{12}^{20} + 0.5h_{20}^{20} \end{cases}.$$

Using substitution we get

$$h_0^{20} = 0, h_4^{20} = \frac{1}{5}, h_8^{20} = \frac{2}{5}, h_{12}^{20} = \frac{6}{10}, h_{16}^{20} = \frac{4}{5}, h_{20}^{20} = 1.$$

(ii) Here we aim at finding the mean hitting time $k_4^A = E[H^A \,|X_0 = 4\$]$, where $A = \{0\$, 20\$\}$. To this end we apply Th. 3.2.6 in the manuscript. The latter amounts to computing the unique solution of the following system of linear equations:

$$\begin{cases} k_i^A = 0, i = 0, 20 \\ k_4^A = 1 + \frac{1}{2}k_8^A \\ k_8^A = 1 + \frac{1}{2}k_{16}^A \\ k_{12}^A = 1 + \frac{1}{2}k_4^A \\ k_{16}^A = 1 + \frac{1}{2}k_{12}^A \end{cases}.$$

This implies that

$$k_4^A = 2 \text{ tosses}.$$

Problem 5

(i) Using the definition of (conditional) expectation we find that

$$\begin{aligned} & E[X_3 \,|X_0 = i] \\ = & \sum_{j=0}^{4} j \cdot P(X_3 = j \,|X_0 = i) \\ & \overset{\text{Th. 3.1.5 in the manuscript}}{=} \sum_{j=0}^{4} j \cdot p_{ij}^{(3)}, \end{aligned}$$

where $p_{ij}^{(3)}$ are the entries of the matrix

$$P^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 81/256 & 108/256 & 54/256 & 12/256 & 1/256 \\ 16/256 & 64/256 & 96/256 & 64/256 & 16/256 \\ 1/256 & 12/256 & 54/256 & 108/256 & 81/256 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

So we obtain

$$E[X_3 \,|X_0 = i] = i \text{ for } i = 0, 1, 2, 3, 4.$$

(ii) Using Th. 3.2.4 in the manuscript (just as in Prob. 2 (i)) we have to determine the hitting probabilities h_i^4, $i \in I = \{0, 1, 2, 3, 4\}$ as the minimal non-negative solution of the linear equation

$$\begin{cases} h_i^4 = 1, & \text{if } i = 4 \\ h_i^4 = \sum_{j=0}^{4} p_{ij} h_j^4, & \text{if } i = 0, 1, 2, 3 \end{cases}.$$

Solving this equation immediately yields

$$h_i^4 = \frac{i}{4} \text{ for } i = 0, 1, 2, 3, 4.$$

Problem 6 We need the following auxiliary result:

Claim: Let $X_n, n \geq 0$ be $Markov(\lambda, P)$ and T a stopping time such that $T < \infty$ with probability 1. Then

$$P(X_T = i_0, X_{T+1} = i_1, ..., X_{T+n} = i_n \,|X_T = i) = P(X_0 = i_0, X_1 = i_1, ..., X_n = i_n \,|X_0 = i) \tag{1}$$

for all $i_0, ..., i_n, i$.

The latter relation follows from the last two lines of the proof of Th. 1.4.2 (Strong Markov property) in Norris. In particular, if $f(x_0, x_1, x_2, ...)$ is a function one obtains from (1) the following useful formula:

$$\begin{aligned} f(X_T, X_{T+1}, X_{T+2}, ...) \text{ conditional } X_T &= i \text{ has the same distribution} \\ \text{as } f(X_0, X_1, X_2...) \text{ conditional on } X_0 &= i. \end{aligned} \tag{2}$$

(i) Because of the definition of the stopping times T_m, $m \geq 0$ we see that $Y_m \in J$ for all m. Since $T_0 < T_1 < .... < T_m < T_{m+1} < \infty$ it follows from the strong Markov property (Th. 1.4.2 in Norris) that

$$X_{T_m+n}, n \geq 0 \text{ is independentof } X_{T_0}, ..., X_{T_m} \text{ conditional on } X_{T_n} = i. \tag{3}$$

By definition we have

$$\begin{aligned} P(Y_{m+1} &= i_{m+1} \,|Y_0 = i_0, ..., Y_m = i_m) \\ &= P(X_{T_{m+1}} = i_{m+1} \,|X_{T_0} = i_0, ..., X_{T_m} = i_m)\,. \end{aligned} \tag{4}$$

Consider the excursion length

$$S \stackrel{def}{=} \inf\{n \geq 1 : X_{T_m+n} \in J\} < \infty.$$

Then

$$T_{m+1} = T_m + S.$$

So we get

$$\begin{aligned} P(X_{T_{m+1}} &= i_{m+1} \,|X_{T_0} = i_0, ..., X_{T_m} = i_m) \\ &= P(X_{T_m+S} = i_{m+1} \,|X_{T_0} = i_0, ..., X_{T_m} = i_m) \\ &= P(X_{T_m+S} = i_{m+1}, S = 1, 2, 3, ... \,|X_{T_0} = i_0, ..., X_{T_m} = i_m) \\ &= \sum_{n\geq 1} P(X_{T_m+S} = i_{m+1}, S = n \,|X_{T_0} = i_0, ..., X_{T_m} = i_m) \\ &= \sum_{n\geq 1} P(X_{T_m+n} = i_{m+1}, S = n \,|X_{T_0} = i_0, ..., X_{T_m} = i_m)\,. \end{aligned} \tag{5}$$

Define the function

$$f(x_0, x_1, x_2, ...) = \begin{pmatrix} x_n \\ \inf\{k \geq 1 : x_k \in J\} \end{pmatrix}. \tag{6}$$

So we see that

$$\begin{aligned} f(X_{T_m}, X_{T_m+1}, X_{T_m+2}, ...) &= \begin{pmatrix} X_{T_m+n} \\ \inf\{k \geq 1 : X_{T_m+k} \in J\} \end{pmatrix} \\ &= \begin{pmatrix} X_{T_m+n} \\ S \end{pmatrix} \end{aligned} \tag{7}$$

Hence because of (3) we must also have that

$$\begin{pmatrix} X_{T_m+n} \\ S \end{pmatrix} \text{ is independent of } X_{T_0}, ..., X_{T_m} \text{ conditional on } X_{T_n} = i.$$

So using the definition of conditional probabilities it follows that

$$\begin{aligned} P(X_{T_m+n} &= i_{m+1}, S=n \,|X_{T_0}=i_0,...,X_{T_m}=i_m) \\ &= P(X_{T_m+n}=i_{m+1}, S=n \,|X_{T_m}=i_m)\,. \end{aligned}$$

Then relation (5) implies that

$$\begin{aligned} P(X_{T_{m+1}} &= i_{m+1} \,|X_{T_0}=i_0,...,X_{T_m}=i_m) \\ &= \sum_{n\geq 1} P(X_{T_m+n}=i_{m+1}, S=n \,|X_{T_m}=i_m) \\ &= P(X_{T_m+S}=i_{m+1}, S=1,2,3,... \,|X_{T_m}=i_m) \\ &= P(X_{T_{m+1}}=i_{m+1} \,|X_{T_m}=i_m) \\ \overset{def}{=} P(Y_{m+1} &= i_{m+1} \,|Y_m=i_m)\,, \end{aligned} \tag{8}$$

which gives (i).

(ii) Because of (8) we can write

$$\begin{aligned} P(Y_{m+1} &= i_{m+1} \,|Y_m=i_m) \\ &= \sum_{n\geq 1} P(X_{T_m+n}=i_{m+1}, S=n \,|X_{T_m}=i_m)\,. \end{aligned}$$

Using formula (2) and (7) we get

$$\begin{aligned} \sum_{n\geq 1} P(X_{T_m+n} &= i_{m+1}, S=n \,|X_{T_m}=i_m) \\ &= \sum_{n\geq 1} P(f(X_0,X_1,X_2...)=\begin{pmatrix} i_{m+1} \\ n \end{pmatrix} |X_0=i_m) \\ &= \sum_{n\geq 1} P(X_n=i_{m+1}, T_1=n \,|X_0=i_m) \\ &= P(X_{T_1}=i_{m+1}, T_1=1,2,3,... \,|X_0=i_m) \\ &= P(X_{T_1}=i_{m+1} \,|X_0=i_m)\,, \end{aligned}$$

if $i_m \in J$.

(iii) a) We want to show that $h_i^j, i\in I$ solves the linear equation

$$h_i^j = p_{ij} + \sum_{k\notin J} p_{ik} h_k^i. \tag{9}$$

To this end let us define the stopping time

$$\widetilde{T} = \inf\{n\geq 1 : X_n \in J\} < \infty.$$

Then

$$\begin{aligned} P(X_{\widetilde{T}} &= j \,|X_0 = i) = P(X_{\widetilde{T}} = j, X_1 \in I \,|X_0 = i) \\ &= \sum_{k\in I} P(X_{\widetilde{T}} = j, X_1 = k \,|X_0 = i) \\ &= \sum_{k\in I} \frac{P(X_{\widetilde{T}} = j, X_1 = k, X_0 = i)}{P(X_1 = k, X_0 = i)} \cdot P(X_1 = k \,|X_0 = i) \\ &= \sum_{k\in I} P_i(X_{\widetilde{T}} = j \,|X_1 = k) \cdot P(X_1 = k \,|X_0 = i) \\ &= \sum_{k\in I} P_i(X_{\widetilde{T}} = j \,|X_1 = k)\, p_{ik}. \end{aligned} \tag{10}$$

Using the definition of $\widetilde{T}$ we observe that

$$P_i(X_{\widetilde{T}} = j \,|X_1 = k) = 0,$$

if $k \in J$ and $k \neq j$. So (10) gives

$$\begin{aligned} P(X_{\widetilde{T}} &= j \,|X_0 = i) = P_i(X_{\widetilde{T}} = j \,|X_1 = j)\, p_{ij} + \sum_{k\notin J} P_i(X_{\widetilde{T}} = j \,|X_1 = k)\, p_{ik} \\ &= p_{ij} + \sum_{k\notin J} P_i(X_{\widetilde{T}} = j \,|X_1 = k)\, p_{ik}. \end{aligned} \tag{11}$$

Claim:

$$P_i(X_{\widetilde{T}} = j \,|X_1 = k) = P(X_{\widetilde{T}} = j \,|X_0 = k) \tag{12}$$

for all $k \notin J$ and $i \in I$.

Proof:

$$\begin{aligned} P_i(X_{\widetilde{T}} &= j \,|X_1 = k) = P(X_{\widetilde{T}} = j \,|X_0 = i, X_1 = k) \\ &= P(X_{\widetilde{T}} = j, \widetilde{T} = 2, 3, ... \,|X_0 = i, X_1 = k) \\ &= \sum_{l\geq 2} P(X_l = j, \widetilde{T} = l \,|X_0 = i, X_1 = k) \\ &= \sum_{l\geq 2} P(X_l = j, \inf\{n \geq 2 : X_n \in J\} = l \,|X_0 = i, X_1 = k) \end{aligned} \tag{13}$$

Since $X_{1+n}, n \geq 0$ is independent of X_0 conditional on $X_1 = k$ we see that X_l and $\inf\{n \geq 2 : X_n \in J\}$ are independent of X_0 conditional on $X_1 = k$ (if $l \geq 2$). So

$$\begin{aligned} P(X_l &= j, \inf\{n \geq 2 : X_n \in J\} = l \,|X_0 = i, X_1 = k) \\ &= P(X_l = j, \inf\{n \geq 2 : X_n \in J\} = l \,|X_1 = k)\,. \end{aligned}$$

Therefore we obtain from (13) that

$$\begin{aligned}
P_i(X_{\widetilde{T}} &= j\,|X_1 = k) \\
&= \sum_{l\geq 2} P(X_l = j, \inf\{n \geq 2 : X_n \in J\} = l\,|X_1 = k) \\
&= \sum_{l\geq 2} P(X_{(l-1)+1} = j, \inf\{n \geq 1 : X_{1+n} \in J\} = l-1\,|X_1 = k) \\
\overset{\text{Formula (2)}}{=} \sum_{l\geq 2} P(X_{l-1} &= j, \inf\{n \geq 1 : X_n \in J\} = l-1\,|X_0 = k) \\
&= \sum_{l\geq 1} P(X_l = j, \inf\{n \geq 1 : X_n \in J\} = l\,|X_0 = k) \\
\overset{\text{Def. of } \widetilde{T}}{=} \sum_{l\geq 1} P(X_l &= j, \widetilde{T} = l\,|X_0 = k) \\
&= P(X_{\widetilde{T}} = j, \widetilde{T} = 1,2,3,...\,|X_0 = k) \\
&= P(X_{\widetilde{T}} = j\,|X_0 = k)\,,
\end{aligned}$$

which yields the claim.

If we combine (11) with (12), we get

$$\begin{aligned}
P(X_{\widetilde{T}} &= j\,|X_0 = i) = p_{ij} + \sum_{k\notin J} P_i(X_{\widetilde{T}} = j\,|X_1 = k)\,p_{ik} \\
&= p_{ij} + \sum_{k\notin J} P(X_{\widetilde{T}} = j\,|X_0 = k)\,p_{ik} \\
&\overset{\text{Def. of } h_k^j}{=} p_{ij} + \sum_{k\notin J} h_k^j p_{ik}
\end{aligned}$$

as desired.

b) Suppose $x_i^j, i \in I$ is a non-negative solution of equation (9). Then one shows just as in the second part of Th. 1.3.2 in Norris that

$$\begin{aligned}
x_i^j &\geq P(X_1 = j\,|X_0 = i) + P(X_1 \notin J, X_2 = j\,|X_0 = i) + ... \\
+P(X_1 &\notin J, ..., X_{n-1} \notin J, X_n = j\,|X_0 = i)
\end{aligned}$$

for all n. Thus

$$\begin{aligned}
x_i^j &\geq \lim_{n\longrightarrow\infty}(P(X_1 = j\,|X_0 = i) + P(X_1 \notin J, X_2 = j\,|X_0 = i) + ... \\
+P(X_1 &\notin J, ..., X_{n-1} \notin J, X_n = j\,|X_0 = i)) \\
&= P(X_{T_1} = j, T_1 = 1,2,3,...\,|X_0 = i) \\
&= P(X_{T_1} = j\,|X_0 = i) \overset{def}{=} h_i^j \text{ (if } i \in J).
\end{aligned}$$