Modeling by Stochastic Processes

1. Introduction

Basic literature :

course books :

1. Norris, J.R. : Markov chains
   Cambridge University Press (1997).
2. Durrett, R. : Essentials of Stochastic Processes.
   3rd edition Springer (2016).

The course deals with the theory of stochastic processes and its applications to modeling of random phenomena in physics, biology, mathmatical finance,

Ex. (modeling of asset/stock prices)

Denote by S_t the price of a stock (e.g. Statoil, Siemens, ...) at time $0 \le t < \infty$ ($t=0$ present time)

→ outcome of S_t at future time $t>0$ is uncertain, i.e.

$S_t = 20\$$ or $25\$$ or $30\$$ ...

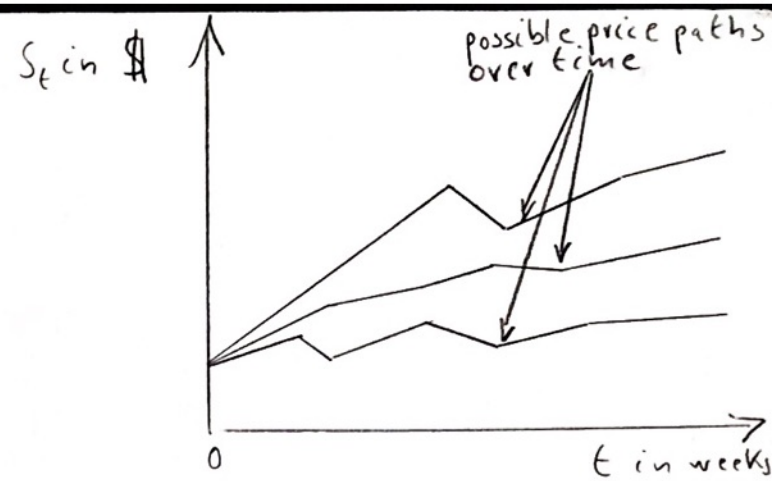
→ The price S_t at $t>0$ is subject to a random source

→ reasonable to model S_t as a random variable (see later) for each $t>0$

→ The stock prices can be described by a collection of random variables (r.v's)

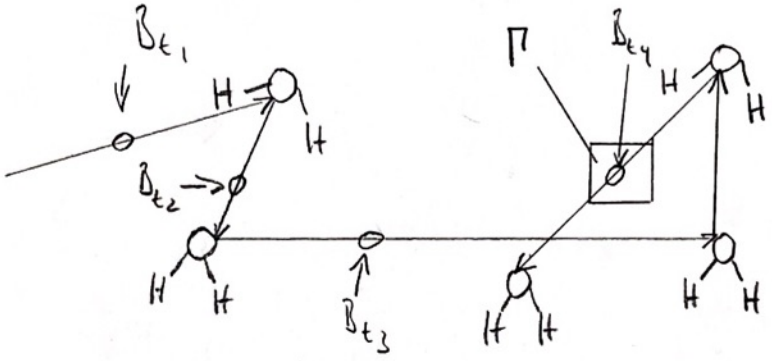
$$\{S_t\}_{0 \le t < \infty},$$

that is by stochastic process



Ex. (diffusion of particles)

Consider the diffusion of a particle in a liquid with position $B_t \in \mathbb{R}^3$ at time $t \geq 0$



$\longrightarrow$ particle is exposed to molecular bombardments

$\longrightarrow$ particle looses its "memory" as a result of collisions

or more mathematically

Probability for $B_t \in \Gamma$ given the knowledge on $B_{t_1}, \dots, B_{t_n}$

$$\text{Prob}\left(B_t \in \Gamma \mid B_{t_1}, \dots, B_{t_n}\right) = \text{Prob}\left(B_t \in \Gamma \mid B_{t_n}\right)$$

$\Gamma \subseteq \mathbb{R}^3$, $0 \leq t_1 < t_2 < \dots < t_n < t$

$\longrightarrow$ particle just remembers its last position B_{t_n}

Useful model for B_t : Brownian motion (see later)

$\longrightarrow$ stochastic process, which is an important example of a Markov process, i.e. a process "without memory"

In this course we shall focus on stochastic processes given by Markov processes

Markov processes form an important class of

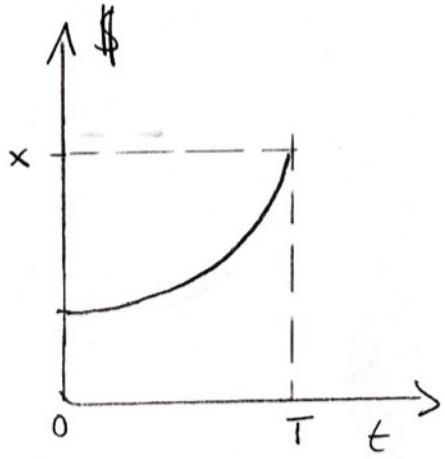
(3) stochastic processes and have e.g. following applications:

1. Black-Scholes model (1973)

Consider e.g. a bond (reutepapir) paying x $ at maturity (utløpstidspunkt) with respect to a fixed interest rate (reute) μ

→ bond price $P(t,T)$ at time $t \leq T$:

$$P(t,T) = x \cdot \exp(-\mu(T-t))$$



→ safe investment

However, the price of a stock S_t at time $t > 0$ cannot be forecasted

→ risky investment

→ reasonable model for S_t:

bond price superposed by a noise

or more generally

exponential growth model distorted by a noise

→

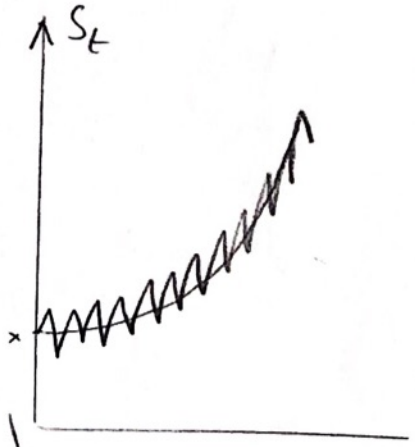
$$S_t = x \cdot \exp(t \cdot \mu + \text{Noise}(t))$$

Noise(t) stoch. proc.

Choose e.g.

$$\text{Noise}(t) = \sigma \cdot B_t,$$

where B_t is a (1-dim.) Brownian motion and $\sigma \geq 0$ a constant (volatility of S_t)



→ celebrated Black-Scholes model for stock prices (Scholes, Merton Nobel price in economics in 1997)

Rem.: (i) One shows that S_t given by the Black-Scholes model is a Markov process

(ii) Actually, empirical studies indicate the "Markovian nature" of stock price data

→ stock prices have no memory

i.e. historical stock prices have no impact on current prices

2. Modeling of the total claim amount:

Modeling of the total claim amount (totalskadesum) in a portfolio of insurance policies (forsikrings-kontrakter) plays a central role in non-life insurance (skadeforsikring) and serves e.g. as a tool for premium calculation or the study of ruin probabilities.

A common model for the total claim amount $S(t)$ at time t is the following (Lundberg 1903):

$$S(t) = \sum_{i=1}^{N(t)} X_i = X_1 + \dots + X_{N(t)}, \quad t \geq 0$$

where

X_i claim size (Kravstørrelse) of a claim (e.g. fire loss) occuring at the claim arrival time T_i, $i \in \mathbb{N}$

$(0 \leq T_1 < T_2 < \dots)$

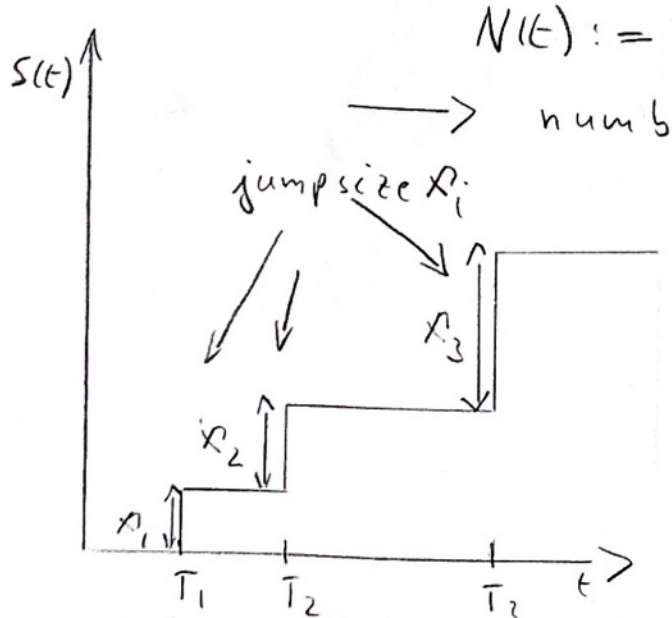
and

$N(t) \in \mathbb{N}_0$ claim numbers given by

$$N(t) := \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0$$

(number of)

→ number of claims arriving by time t



(5) Under reasonable assumptions on the r.v.'s T_i, X_i $S(t)$ becomes a Markov process given e.g. by a compound Poisson process (see later)

3. Option pricing in finance:

Consider an option h given e.g. by a European call option, that is the right (but not obligation) to buy a certain amount of financial assets (e.g. stocks) by a certain date T (maturity) for a certain strike price K

→ payoff (uitbetalingsfunksjon) at T:

$$h = (S_T - K)^+ := \max(S_T - K, 0)$$

S_T e.g. stock price at maturity T

A more complex example is a path dependent option as e.g. the knock out option of the form

$$h = 1_{[K,\infty)}\left(\max_{0 \le t \le T} S_t\right)$$

$$= \begin{cases} 1\ (\$), & K \le \max\limits_{0 \le t \le T} S_t \\ 0\ (\$) & \text{else} \end{cases}$$

→ Black, Scholes (1973)

What is the fair value of an option?

→ If e.g. S_t is described by the Black-Scholes model in 1., fair pricing based on the dynamic hedging principle gives

$$\text{OptionValue}_t = E\left[\underbrace{e^{-r(T-t)}}_{\text{discount factor}} \cdot h \mid \mathcal{F}_t\right] \quad (*)$$

$\triangleq$ expected discounted option h given the entirety of market information up to time t (i.e. $\mathcal{F}_t$)

in a "risk-neutral world" (see later)

r (constant) market interest rate

<u>In general</u>, (*) is difficult to compute.

<u>However</u>, if S_t is a Markov process, (*) can be calculated in many cases