all states are periodic of period 2

Ex. 3.6.4 : Car insurance of Ex. 3.2.7

$\longrightarrow$ X_n risk category of the customer in year n with

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0.6 & 0.4 \end{pmatrix}$$



Assumption : $P(X_0 = i) > 0$ for all $i = 1,2,3$

Let us check $i = 1$:

Suppose n even, i.e. $n = 2m$ for $m \geq 1$

$$\Rightarrow \quad p_{11}^{(n)} = P(X_n = 1 \text{ "and" } | X_0 = 1) =$$

$$P(X_0 \in \{1,2,3\}, X_1 \in \{1,2,3\}, \dots, X_{n-1} \in \{1,2,3\}, X_n = 1 \mid X_0 = 1)$$

$$= \sum_{i_0=1}^{3} P(X_0 = i_0, X_1 \in \{1,2,3\}, \dots, X_{n-1} \in \{1,2,3\}, X_n = 1 \mid X_0 = 1)$$

$$= \sum_{i_0=1}^{3} \cdots \sum_{i_{n-1}=1}^{3} \underbrace{P(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = 1 \mid X_0 = 1)}_{= 0, \text{ if } i_0 \neq 1}$$

Th. 3.1.3

$$= \sum_{i_1=1}^{3} \cdots \sum_{i_{n-1}=1}^{3} p_{1i_1} p_{i_1 i_2} \cdots p_{i_{n-1} 1}$$

$$\geq p_{12} p_{21} p_{12} \cdots p_{21} > 0 \qquad n = 2m$$

$\leftarrow$ i.e. the chain moves m times forth and back between 1 and 2

Similarly one sees

$p_{11}^{(n)} > 0$ for n odd

$$\Rightarrow \quad p_{11}^{(n)} > 0 \text{ for all } n \geq 1$$

$$\Rightarrow d(1) = 1$$

Since all states communicate, it follows from Lemma 3.6.3 that

$$d(i) = 1, \quad i = 1,2,3$$

$\Rightarrow$ all states are aperiodic

Theorem 3.6.5 ((convergence to equilibrium)) Let $X_n, n \geq 0$ be Markov(λ, P) with state space $\mathcal{I}$. Suppose that $\mathcal{I}$ is the only communicating class (i.e. $X_n, n \geq 0$ is irreducible). Further require that all $i \in \mathcal{I}$ are aperiodic and that $X_n, n \geq 1$ has an invariant distribution Π.

(λ: initial distribution)

Then
$$P(X_n = j) \longrightarrow \pi_j \quad \text{as } n \to \infty \text{ for all } j$$
In particular, since the initial distr. $\lambda = (\lambda_i)_{i \in I}$ is arbitrary, we have
$$p_{ij}^{(n)} \longrightarrow \pi_j \quad \text{as } n \to \infty \text{ for all } i, j$$

Ex. 3.6.6 : Car insurance of Ex. 3.6.4

We know from Ex. 3.6.4 that $X_n, n \geq 0$ is irreducible and aperiodic

Because of Ex. 3.5.8 $X_n, n \geq 0$ has a (unique) invariant distr., i.e. there ex. $\pi = (\pi_1, \pi_2, \pi_3)$ s.t.
$$\pi \cdot P = P \text{ and } \pi_1 + \pi_2 + \pi_3 = 1$$
Ex 3.5.8 $\longrightarrow$ $\pi_1 = 3\pi_2, \pi_2 = 1.2 \cdot \pi_3$ with $\pi_3 = \frac{1}{5.8} \approx 0.17$

$\Rightarrow$ $\pi_1 = \frac{3.6}{5.8} \approx 0.62, \pi_2 = 0.21$

Th. 3.6.5 $\Rightarrow$ $P(X_n = i) \xrightarrow[n \to \infty]{} \pi_i$

$\Rightarrow$ e.g. the prob. to reach the highest risk category amounts to 17% in the long run

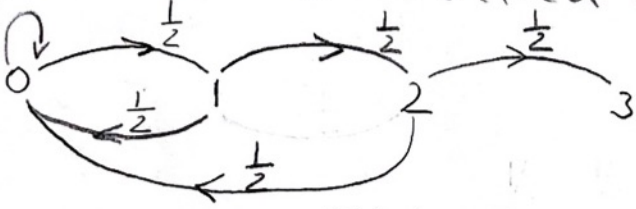
Ex. 3.6.7 (Success Runs) See Exerc. 7, Prob. 3.

Fair coin flipped repeatedly, X_n the length of the current run of heads at time $n \geq 0$

E.g. H T H H H T ($X_0 \equiv 0$)

$\Rightarrow$ $X_1 = 1, X_2 = 0, X_3 = 1, X_4 = 2, X_5 = 3, X_6 = 0$

$\longrightarrow$ $X_n, n \geq 0$ can be modelled by the Markovchain



$\longrightarrow$ $X_n, n \geq 0$ irreducible

Find an invariant distr. π : $\pi \cdot P = \pi$ becomes

$$(\pi_0, \pi_1, \dots) \cdot \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 1/2 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 1/2 & 0 & \dots \\ 1/2 & 0 & 0 & 0 & 1/2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \leftarrow P = (\pi_0, \pi_1, \dots)$$

$\longrightarrow$ solution: $\pi_k = \frac{1}{2^{k+1}}, k = 0, 1, 2, \dots$

Recall: geom. sum: $\sum_{k \geq 0} (q)^n = \frac{1}{1-q}, 0 < q < 1$

$q = \frac{1}{2}$ $\Rightarrow$ $\sum_{k \geq 0} \pi_k = \sum_{k \geq 0} \frac{1}{2^{k+1}} = \frac{1}{2}\left(\sum_{k \geq 0} \frac{1}{2^k}\right) = \frac{1}{2} \cdot 2 = 1$

(60) $\Rightarrow \exists!$ inv. distr.

Just as in Ex. 3.6.4 one checks that $p_{00}^{(n)} > 0$ for all n

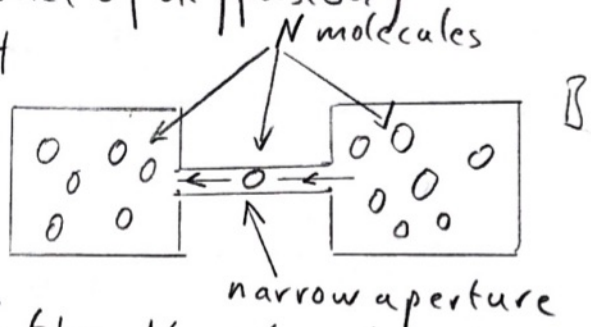
$\Rightarrow$ 0 aperiodic $\overset{L.3.6.3}{\Rightarrow}$ all i are aperiodic

$\overset{Th.3.6.5}{\Rightarrow}$ $P(X_n = i) \xrightarrow[n\to\infty]{} \pi_i = \frac{1}{2^{i+1}}$

$\overset{i=1}{\Rightarrow}$ prob. that the length of the current run of heads equals 1 is 25% in the long run

Ex. 3.6.8 (Ehrenfest model of diffusion)

See Exerc. 7, Prob. 4



Each time unit 1 of the N molecules is chosen at random to pass through the narrow aperture

$\longrightarrow$ X_n denotes the number of molecules in container A at time $n \geq 0$

$\longrightarrow$ Markov model:



$\Longrightarrow$ $X_n, n \geq 0$ Markov chain with $S = \{0, \dots, N\}$ and

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{N} & 0 & \frac{N-1}{N} & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{N} & 0 & \frac{N-2}{N} & 0 & \cdots & 0 \\ 0 & 0 & \frac{3}{N} & 0 & \frac{N-3}{N} & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \; 0 \end{pmatrix}$$

$\hat{\pi} \cdot P = \hat{\pi}$ for $\hat{\pi} = (\hat{\pi}_0, \dots, \hat{\pi}_N)$ becomes

$$\begin{cases} \hat{\pi}_1 \cdot \frac{1}{N} = \hat{\pi}_0 \\ \hat{\pi}_0 + \hat{\pi}_2 \frac{2}{N} = \hat{\pi}_1 \\ \hat{\pi}_1 \frac{N-1}{N} + \hat{\pi}_3 \frac{3}{N} = \hat{\pi}_2 \\ \vdots \\ \hat{\pi}_{N-2} \frac{1}{N} + \hat{\pi}_N \frac{N}{N} = \hat{\pi}_{N-1} \end{cases}$$

i.e. $\hat{\pi}_{K-1} \frac{N-K}{N} + \hat{\pi}_{K+1} \frac{K+1}{N} = \hat{\pi}_K \quad , K = 1, \dots, N-1$

with $\hat{\pi}_1 \cdot \frac{1}{N} = \hat{\pi}_0$

Suppose $\hat{\pi}_0 = A \geq 0 \Longrightarrow \hat{\pi}_1 = N \cdot A = \binom{N}{1} A$ $\quad \left| \binom{N}{K} := \frac{N!}{(N-K)!\,K!} \right.$

$\Longrightarrow$ $A + \hat{\pi}_2 \frac{2}{N} = \binom{N}{1} A \Longrightarrow \hat{\pi}_2 = \underbrace{\frac{N(N-1)}{2}}_{= \binom{N}{2}} \cdot A = \binom{N}{2} A$

Or more generally by induction:

$\hat{\pi}_k = \binom{N}{k} \cdot A \;,\; k = 0, \dots, N$

$\hat{\pi}$ inv. distr. if $\sum_{k=0}^{N} \hat{\pi}_k = 1 \iff$

$\sum_{k=0}^{N} \binom{N}{k} \cdot A = 1$

e.g. $\sum_{k=0}^{3} \binom{3}{k} = 1 + 3 + 3 + 1 = 8 = 2^3$

Or more generally

$\sum_{k=0}^{N} \binom{N}{k} = 2^N$

$\Rightarrow A = \frac{1}{2^N} \Rightarrow \hat{\pi}_k = \binom{N}{k} \cdot \frac{1}{2^N} = \binom{N}{k} \left(\frac{1}{2}\right)^{N-k} \cdot \left(\frac{1}{2}\right)^k$

$\longrightarrow$ <u>binomial distr.</u> with param. N and $\frac{1}{2}$

$X_n, n \geq 0$ irreducible

If n even then

$p_{00}^{(n)} = P(X_n = 0 \mid X_0 = 0) \geq P(X_1 = 1, X_2 = 0, X_3 = 1, \dots, X_{n-1} = 1, X_n = 0 \mid X_0 = 0)$

$\overset{\text{Th. 3.1.3}}{=} p_{01} \cdot p_{10} \cdot p_{01} \cdots p_{10} > 0$

On the other hand we have that

$p_{00}^{(n)} = 0$

for odd n, since one cannot come back to 0 after an odd number of steps (see diagram)

$\Rightarrow d(0) = 2 \overset{\text{L. 3.6.3}}{\Longrightarrow} d(i) = 2$ for all $i = 0, \dots, N$

$\Rightarrow X_n, n \geq 0$ is <u>not</u> aperiodic $\Rightarrow$ Th. 3.6.5 is not (directly) applicable

<u>However</u> convergence in equilibrium is <u>still</u> valid in the following sense (see Th. 1.8.5 in Norris)

and $P(X_{2n} = j) \xrightarrow{n \to \infty} 2\hat{\pi}_j$, if j is even

$P(X_{2n+1} = j) \xrightarrow{n \to \infty} 2\hat{\pi}_j$, if j is odd

<u>Provided</u> that the init. distr $\lambda = (\lambda_i)_{0 \leq i \leq N}$ satisfies $\sum_{i \text{ even}} \lambda_i = 1$

<u>Proof of Th. 3.6.5</u> (Conv. to equilibrium)

<u>Idea</u>: <u>coupling argument</u> in the following sense:

Consider <u>another</u> Markov chain $Y_n, n \geq 0$ with the <u>same</u> trans. matrix P, <u>but</u> initial distr. $\hat{\pi}$ (i.e. inv. distr. of $X_n, n \geq 0$)

<u>Assumption</u>: $Y_n, n \geq 0$ indep. of $X_n, n \geq 0$

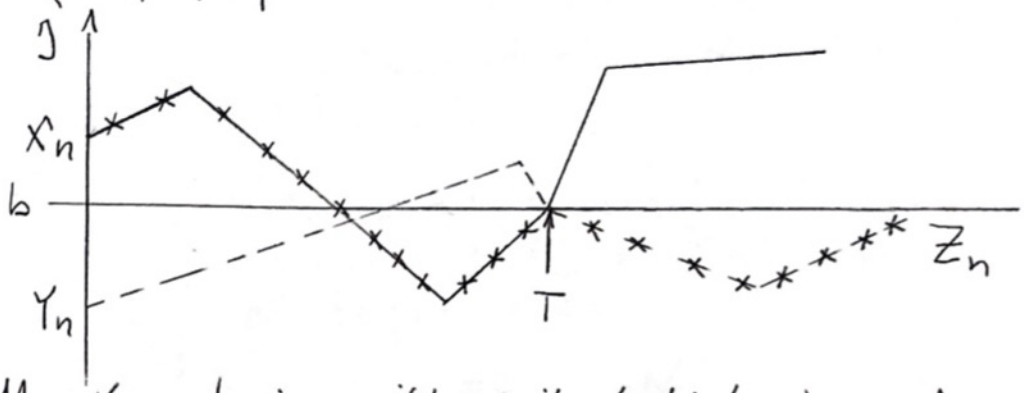
Fix a $b \in \mathbb{N}$. Define the <u>stopping time</u>

$T \overset{\text{def}}{=} \inf\{n \geq 1 : X_n = Y_n = b\}$

Norris, P. 41 $\rightarrow$ $T < \infty$ with prob. 1

(63) Further define the chain

$$Z_n = \begin{cases} X_n, & \text{if } n < T \\ Y_n, & \text{if } n \geq T \end{cases}$$



Norris P. 42 $\rightarrow$ $Z_{n,n\geq 0}$ Markov chain with initial distr. λ and trans. matrix P (as $X_{n,n\geq 0}$)

$$P(Z_n = j) = \sum_{i\in J} P(Z_n = j, X_0 = i) \overset{\text{def. of cond. exp.}}{=} \sum_{i\in J} P(Z_n = j \mid \underbrace{Z_0 = i}_{=X_0=i}) \cdot \underbrace{\lambda_i}_{=P(X_0=i)}$$

$$\overset{\text{Th. 3.1.3}}{=} \sum_{i\in J} p_{ij}^{(n)} \cdot \lambda_i \overset{\text{Th. 3.1.3 (i)}}{=} P(X_n = j)$$

$$P(Y_n = j) \overset{\text{Th. 3.1.3}}{=} \sum_{i\in J} p_{ij}^{(n)} \cdot \pi_i \overset{\pi \text{ inv. distr.}}{=} \pi_j$$

$$\Rightarrow \quad |P(X_n = j) - \pi_j| = |P(Z_n = j) - P(Y_n = j)| \qquad (+)$$

$$P(Z_n = j) = P(\{X_n = j \text{ and } n < T\} \overset{\text{disjoint}}{\cup} \{Y_n = j \text{ and } n \geq T\})$$

by def. of Z_n

$$\Rightarrow P(Z_n = j) = P(X_n = j, \overset{\text{"and"}}{n < T}) + P(Y_n = j, n \geq T)$$

$$\overset{(+)}{\Rightarrow} |P(X_n = j) - \pi_j| = |P(X_n = j, n < T) + P(Y_n = j, n \geq T) - P(Y_n = j)| \qquad (++)$$

$$P(Y_n = j) = P(\{Y_n = j, n \geq T\} \cup \{Y_n = j, n < T\})$$

$$= P(Y_n = j, n \geq T) + P(Y_n = j, n < T)$$

$$\overset{(++)}{\Rightarrow} |P(X_n = j) - \pi_j| = |P(X_n = j, n < T) + P(Y_n = j, n \geq T) - P(Y_n = j, n \geq T) - P(Y_n = j, n < T)|$$

$$= |P(X_n = j, n < T) - P(Y_n = j, n < T)|$$

$$\leq \underbrace{P(X_n = j, n < T)}_{\leq P(n<T)} \text{ or } \underbrace{P(Y_n = j, n \geq T)}_{P(n<T)}$$

$$\leq P(n < T) \xrightarrow[n\to\infty]{} 0 \quad \text{since } T < \infty \text{ with prob. 1}$$

$\Rightarrow$ proof