$\Rightarrow$ countable contradiction (since states $i=1,2,3,\dots$ transient)

$\lim_{n\to\infty} X_n$, $\overline{\lim_{n\to\infty}} X_n = 0$ or ∞

$\Rightarrow$ "trap" with prob. 1

either $\lim_{n\to\infty} X_n = 0$ or $\lim_{n\to\infty} X_n = \infty$

$\rightarrow$ either the number of family names disappears or explodes

Problem: $P(X_n = 0 \text{ for some } n)$?

Consider the probability generating function:

$$\mathcal{G}_Y(t) \overset{\text{def}}{=} E[t^Y] = \sum_{k\geq 0} t^k P(Y=k), \quad 0 \leq t \leq 1 \qquad (4.1.3)$$

$\rightarrow$ can be used to compute moments of r.v.'s $Y \in \{0,1,2,\dots\}$

$$E[t^{X_2} \mid X_1 = k] \overset{\text{def of}}{=} E[t^{N_1^2 + \dots + N_{X_1}^2} \mid X_1 = k] \overset{X_n}{=} E[t^{N_1^2 + \dots + N_k^2} \mid X_1 = k] = E[t^{N_1^2} \cdots t^{N_k^2}]$$

($N_1^2, \dots, N_k^2$ indep.; $t^{N_1^2}, \dots, t^{N_k^2}$ indep.)

$$= E[t^{N_1^2}] \cdots E[t^{N_k^2}] \overset{N_i^n \text{ i.i.d}}{=} E[t^N] \cdots E[t^N] \overset{\text{def}}{=} (\phi(t))^k$$

($E[t^N] = \mathcal{G}_N(t) \overset{\text{def}}{=} \phi(t)$)

$$\Rightarrow E[t^{X_2}] = \sum_{k\geq 0} E[t^{X_2} \cdot \mathbb{1}_{\{X_1 = k\}}] = \sum_{k\geq 0} E[t^{X_2} \mid X_1 = k] \cdot P(X_1 = k) \overset{(*)}{=} \sum_{k\geq 0} (\phi(t))^k \cdot P(X_1 = k)$$

$$\overset{\text{def. of expect.}}{=} E[(\phi(t))^{X_1}] \overset{X_1 = N}{=} E[(\phi(t))^N] \overset{\text{def}}{=} \phi(u)\big|_{u=\phi(t)} = \phi(\phi(t)) = \phi^{(2)}(t)$$

Or more generally

$$\mathcal{G}_{X_n}(t) \overset{\text{def}}{=} E[t^{X_n}] = \phi^{(n)}(t) \quad (= \underbrace{\phi \circ \dots \circ \phi}_{n \text{ times}}(t)) \qquad (4.1.4)$$

$$\frac{d}{dt}\mathcal{G}_{X_n}(t) \overset{(4.1.3)}{=} \sum_{k\geq 1} k P(X_n = k) t^{k-1}$$

$$\Rightarrow E[X_n] = \lim_{t\uparrow 1} \frac{d}{dt}\mathcal{G}_{X_n}(t) \overset{(4.1.4)}{=} \lim_{t\uparrow 1} \frac{d}{dt}(\phi \circ \dots \circ \phi(t)) =$$

$$\left(\lim_{t\uparrow 1} \frac{d}{dt}\phi(t)\right)^n = \left(\lim_{t\uparrow 1} \frac{d}{dt}\mathcal{G}_N(t)\right)^n = \left(\underbrace{E[N]}_{\overset{\text{def}}{=}\mu}\right)^n \qquad (4.1.5)$$

From (4.1.4) it also follows for $t=0$ that

$$P(X_n = 0) = \phi^{(n)}(0) \quad (= \phi \circ \dots \circ \phi(0)) \qquad (4.1.6)$$

$\Rightarrow$ "trap"

$$P(X_n = 0 \text{ for some } n) = P(\lim_{n\to\infty} X_n = 0)$$

$$= \lim_{n\to\infty} P(X_n = 0) \overset{(4.1.6)}{=} \lim_{n\to\infty} \phi^{(n)}(0) \qquad (4.1.7)$$

Theorem 4.1.1: Def. $q = P(X_n = 0 \text{ for some } n)$

Then the extinction prob. q is the smallest number $\in [0,1]$ s.t.

$$\phi(q) = q$$

Ex. 4.1.2 (population of cells)

Consider a population of cells that may either die or reproduce according to

$$N = \begin{cases} 0 & \text{with probability } \frac{1}{4} \\ 2 & \text{with prob. } \frac{3}{4} \end{cases}$$

$\| X_1$

$\longrightarrow$ X_n number of cells in gen. n

$\Rightarrow$ $\phi(t) \overset{\text{def}}{=} \beta_N(t) \overset{\text{def}}{=} E[t^N] = \sum_{k \geq 0} t^k P(N=k) = \frac{1}{4} + \frac{3}{4} t^2$

$\Rightarrow$ $\phi(t) = t \iff \frac{1}{4} + \frac{3}{4} t^2 = t \iff t = \frac{1}{3}$ or 1

$\overset{\text{Th. 4.1.1}}{\Rightarrow}$ extinction prob. $q = \min\left(\frac{1}{3}, 1\right) = \frac{1}{3} \approx 33\%$

Proof of Th. 4.1.1: Let q be the extinction prob. i.e.

$$q = P(X_n = 0 \text{ for some } n)$$

Define

$$\tau = \min\{t \in [0,1] : \phi(t) = t\}$$

where

$$\phi(t) \overset{\text{def}}{=} \mathcal{P}_N(t) \overset{\text{def}}{=} \sum_{K \geq 0} t^K P(N=K)$$

ϕ continuous $\Rightarrow$ $\phi(\tau) = \tau$

$\tau \geq 0$, ϕ increasing $\Rightarrow$ $\phi(\tau) \geq \phi(0) \Rightarrow \tau \geq \phi(0)$

ϕ increasing $\Rightarrow$ $\tau = \phi(\phi(\tau)) \overset{\tau}{\geq} \phi(\phi(0)) \overset{\text{def}}{=} \phi^{(2)}(0)$ ...

$\Rightarrow \tau \geq \phi \circ \dots \circ \phi(0) \overset{\text{def}}{=} \phi^{(n)}(0)$ for all n

Because of (4.1.7) we know that

$$q = \lim_{n \to \infty} \phi^{(n)}(0)$$

$$\Rightarrow \tau \geq \lim_{n \to \infty} \phi^{(n)}(0) = q$$

On the other hand

$$q = \lim_{n \to \infty} \phi^{(n+1)}(0) = \lim_{n \to \infty} \phi(\phi^{(n)}(0)) = \phi(q)$$

def. of τ $\Rightarrow$ $q \geq \tau \Rightarrow q = \tau \Rightarrow$ proof

Theorem 4.1.2: The survival probability $p \overset{\text{def}}{=} 1-q = P(X_n > 0 \text{ for all } n)$ is strictly bigger than zero if and only if

$$\mu \overset{\text{def}}{=} E[N] > 1$$

Proof: 1. case: $\phi'(1) > 1$

a) $\phi(0) = 0 \overset{\text{Th } 4.1.1}{\Rightarrow} q = 0 < 1$

b) $\phi(0) > 0$

Suppose $\phi(s) - s \neq 0$ for all $0 \leq s < 1$

ϕ continuous $\Rightarrow$ $\phi(s) - s > 0$ for all $0 \leq s < 1$

$\phi'(1) > 1 \Rightarrow \phi'(1) - 1 > 0 \Rightarrow \phi'(s) - 1 > 0$ for all $s \in (1-\varepsilon, 1] \Rightarrow \phi(s) - s$ strictly increasing on $(1-\varepsilon, 1)$

However $\phi(1) \overset{\text{def}}{=} \mathcal{P}_N(1) = \sum_{K \geq 0} 1^K \cdot P(N=K) = 1$

$\Rightarrow \phi(1) - 1 = 0$

$\Rightarrow$ contradiction $\Rightarrow$ there ex. $s \in [0,1)$ s.t. $\phi(s) - s = 0 \overset{\text{Th } 4.1.1}{\Rightarrow} q < 1$

2. case: $\phi'(1) \leq 1$

a) $P(N \geq 2) > 0$:

(77) $\Rightarrow \phi''(s) = \sum_{k\geq 2} k(k-1)\, s^{k-2}\, P(N=k) > 0$

for all $0 < s < 1$

$\Rightarrow$ $\phi'(s) - 1$ strictly increasing on $(0,1)$

$\Rightarrow$ $\phi'(s) - 1 < \phi'(1) - 1 \leq 0$ on $(0,1)$

$\Rightarrow$ $\phi(s) - s$ strictly decreasing on $(0,1)$

$\Rightarrow$ $\phi(1) - 1 = 0$ $\Rightarrow$ $q = 1$

b) $P(N \geq 2) = 0$: $\overset{(4.1.2)}{\Rightarrow}$ $P(N=0) > 0$

and $\phi(s) = P(N=0) + s\, P(N=1)$

$\Rightarrow$ $\phi(s) = s \iff s = 1 \Rightarrow q = 1$ $\Rightarrow$ proof

Ex. 4.1.3 : Exercises II, Prob. 2 :

an individual has an infectious disease, which remains infectious for one day

$\rightarrow$ $X_0 = 1$ numb. of infected individ. at $n=0$

The individual interacts with Z other people

$\rightarrow$ $X_1 = N \overset{\text{def}}{=} \sum_{i=1}^{Z} B_i$ $\leftarrow$ indep. number of infected indiv. at $n=1$

where $Z \sim \text{Pois}(\lambda)$, $\lambda = 10$ and B_i, $i \geq 0$ indep. and binomially distr. with param. $n=1$ and $\tilde{p}$

For which $\tilde{p}$ will the disease disappear ?

Because of Th. 4.1.2 we have to ensure that

$$\mu \overset{\text{def}}{=} E[N] \leq 1$$

$$E[N] = E\Big[\sum_{i=1}^{Z} B_i\Big] = \underbrace{E[Z]}_{=10} \cdot \underbrace{E[B_1]}_{=\tilde{p}} = 10 \cdot \tilde{p}$$

$\Rightarrow$ $\mu \leq 1 \iff \tilde{p} \leq \frac{1}{10} = 10\%$

$P(X_2 \neq 0) \geq$ for $\tilde{p} = 20\%$

$P(X_2 \neq 0) = 1 - P(X_2 = 0)$

Because of (4.1.6) we know that

$P(X_2 = 0) = \phi^{(2)}(0) = \phi(\phi(0))$ (*)

$\phi(t) = E[t^N] = \sum_{k\geq 0} t^k\, P(N=k)$

$$P(N=k) = \sum_{n\geq 0} P(N=k, Z=n) = \sum_{n\geq 0} P\Big(\sum_{i=1}^{n} B_i = k, Z=n\Big)$$

($Z=n$: indep.)

$$= \sum_{n\geq 0} P\Big(\sum_{i=1}^{n} B_i = k\Big) \cdot P(Z=n)$$

(78) $= \sum_{n\geq 0} \binom{n}{k} \tilde{p}^k \tilde{p}^{n-k} \cdot e^{-10} \cdot \frac{10^n}{n!}$

$\Rightarrow \phi(t) = \sum_{k\geq 0} \sum_{n\geq 0} t^k \binom{n}{k} \tilde{p}^k (1-\tilde{p})^{n-k} e^{-10} \cdot \frac{10^n}{n!}$

$= \sum_{n\geq 0} \Big(\underbrace{\sum_{k\geq 0} \binom{n}{k} \Big(t \cdot \frac{\tilde{p}}{1-\tilde{p}}\Big)^k}_{\text{binomial formula } \Big(t\cdot\frac{\tilde{p}}{1-\tilde{p}}+1\Big)^n} \Big) \frac{((1-\tilde{p})\cdot 10)^n}{n!} e^{-10}$

$= \sum_{n\geq 0} \frac{\Big(\Big(t\cdot\frac{\tilde{p}}{1-\tilde{p}}+1\Big)((1-\tilde{p})\cdot 10)\Big)^n}{n!} \cdot e^{-10} \overset{\tilde{p}=0.2}{=} \sum_{n\geq 0} \frac{(2t+8)^n}{n!} e^{-10}$

$= e^{2t+8} e^{-10} = e^{2(t-1)}$

$\Rightarrow \phi(0) = e^{-2} \Rightarrow \phi(\phi(0)) = e^{2(e^{-2}-1)} \approx 0.18$

$\overset{(*)}{\Rightarrow} P(X_2 = 0) \approx 0.18 \Rightarrow P(X_2 \neq 0) \approx 82\%$

4.2 Markov chains in resource management

Ex. 4.2.1 (Restocking a warehouse)
Consider a warehouse with a capacity of c units of stock
Denote by D_n the demand for the stock in units in time period n

$\longrightarrow$ X_n residual stock at the end of period n

The manager restocks to capacity c at the beginning of period $n+1$, if

$X_n \leq m$

where $m \in \{0, \dots, c-1\}$ is a threshold

$$\longrightarrow X_{n+1} = \begin{cases} (c - D_{n+1})^+ & \text{if } X_n \leq m \\ (X_n - D_{n+1})^+ & \text{if } m < X_n \leq c \end{cases} \qquad (4.2.1)$$

$(a)^+ \overset{\text{def}}{=} \max(0, a)$

Assumption: $D_i, i \geq 1$ i.i.d with $D_i \in \{0,1,2,\dots\}$ for all i and common law D

$\Rightarrow$ $X_n, n\geq 0$ Markov chain with $S = \{0, \dots, c\}$, since

$P(X_{n+1} = i_{n+1} \mid X_0 = i_0, \dots, X_n = i_n) =$

$P(X_{n+1} = i_{n+1}, X_n \leq m \text{ or } m < X_n \leq c \mid X_0 = i_0, \dots, X_n = i_n)$

(79)
$$= P(X^?_{n+1} = i_{n+1},\ X^?_n \le m \mid X^?_0 = i_0, \dots, X^?_n = i_n)$$
$$+ P(X^?_{n+1} = i_{n+1},\ m < X^?_n \le c \mid X^?_0 = i_0, \dots, X^?_n = i_n)$$
(4.2.1)
$$= P((c - D_{n+1})^+ = i_{n+1},\ i_n \le m \mid X^?_0 = i_0, \dots, X^?_n = i_n)$$
$$+ P((i_n - D_{n+1})^+ = i_{n+1},\ m < i_n \le c \mid X^?_0 = i_0, \dots, X^?_n = i_n)$$
$$= P((c - D_{n+1})^+ = i_{n+1},\ i_n \le m)$$
$$+ P((i_n - D_{n+1})^+ = i_{n+1},\ m < i_n \le c)$$
$D_i \overset{d}{=} D$
$$= P((c - D)^+ = i_{n+1},\ i_n \le m)$$
$$+ P((i_n - D)^+ = i_{n+1},\ m < i_n \le c)$$
$$= p_{i_n i_{n+1}},$$

where

$$p_{ij} = P((c - D)^+ = j,\ i \le m) + P((i - D)^+ = j,\ m < i \le c) \quad (4.2.2)$$

Recall: A, B indep. $\Rightarrow P(A \mid B) = P(A)$

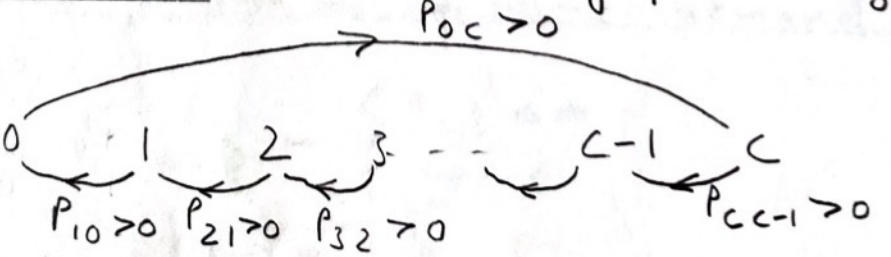
Similarly, one finds

$$P(X^?_{n+1} = i_{n+1} \mid X^?_n = i_n) = p_{i_n i_{n+1}}$$

$\Rightarrow$ $X^?_n, n \ge 0$ Markov chain

Assumption: Choose $D_i, i \ge 1$ such that $X^?_n, n \ge 0$ is irreducible (i.e. $i \leftrightarrow j$ for all $i, j \in \mathbb{N}$)

as e.g.



with $D \sim \text{Pois}(\lambda)$, $\lambda > 0$

Every finite closed class is recurrent (Rem. 3.4.6)

$\Rightarrow$ $X^?_n, n \ge 0$ recurrent and irreducible

4.3.5.10 $\Rightarrow$ $X^?_n, n \ge 0$ has an invariant distribution $\pi_i, i = 0, \dots, c$ (Choose $\pi_i = \frac{\gamma_i^K}{\gamma_0^K + \dots + \gamma_c^K}$ for a fixed $K \in \{0, \dots, c\}$)

Th. 3.5.7 $\Rightarrow$ all states of $X^?_n, n \ge 0$ are positive recurrent

and $m_i = \frac{1}{\pi_i}$, $i = 0, \dots, C$

expected return time to i

$X_n \le m$, $D_{n+1} > C \Rightarrow$ unmet demand at $n+1$ given by

$$0 < D_{n+1} - C = (D_{n+1} - C)^+$$

$m < X_n \le C$, $D_{n+1} > X_n \Rightarrow$ unmet demand at $n+1$ given by

$$0 < D_{n+1} - X_n = (D_{n+1} - X_n)^+$$

Define $\tilde{X}_{n+1} = \begin{cases} (D_{n+1} - C)^+ , & X_n \le m \\ (D_{n+1} - X_n)^+ , & m < X_n \le C \end{cases}$

$\longrightarrow$ $\tilde{X}_{n+1}$ unmet demand at $n+1$

$\longrightarrow$ $E[\tilde{X}_{n+1} \mid X_n] \overset{\text{def}}{=} \underbrace{E[\tilde{X}_{n+1} \mid X_n = i]}_{\overset{\text{def}}{=} f(i)}\Big|_{i = X_n}$ (4.2.3)

$= f(X_n)$

$\longrightarrow$ expected unmet demand in period $n+1$ given the residual stock X_n

Th. 3.8.2 (Ergodic Theorem)

$$\frac{1}{n} \sum_{j=0}^{n-1} E[\tilde{X}_{j+1} \mid X_j] = \frac{1}{n} \sum_{j=0}^{n-1} f(X_j) \xrightarrow{n \to \infty} \bar{f}$$

with $\bar{f} \overset{\text{def}}{=} \sum_{i=0}^{C} \pi_i f(i)$, where

$$f(i) \overset{\text{def}}{=} E[\tilde{X}_{n+1} \mid X_n = i] = \begin{cases} E[(D - C)^+] , & i \le m \\ E[(D - i)^+] , & m < i \le C \end{cases}$$

$\longrightarrow$ $\bar{f}$ long-run proportion of unmet demand

$\bar{f} = \bar{f}(m)$

Def. $g(i) = 1_{\{i \le m\}} = \begin{cases} 1 , & i \le m \\ 0 \text{ else} \end{cases}$

Ergodic Theorem

$$\frac{1}{n} \sum_{j=0}^{n-1} g(X_j) \xrightarrow{n \to \infty} \bar{g} ,$$

where $\bar{g} \overset{\text{def}}{=} \bar{g}(m) \overset{\text{def}}{=} \sum_{i=0}^{C} \pi_i g(i) = \sum_{i=0}^{m} \pi_i$

$\longrightarrow$ $\bar{g}(m)$ long run frequency of restocking

$\longrightarrow$ Objective of the manager : minimization of the (lost) profit of the long-run proportion of unmet demand and the costs for long-run frequency of restocking with respect to the threshold m

(81) $\longrightarrow$ $a \cdot \bar{g}(m) + b \cdot \bar{f}(m) \longrightarrow \min$ in m

$a \triangleq$ cost of restocking per unit stock

$b \triangleq$ profit per unit stock

Ex.: $b = 1\$$, $C = 3$ $\Rightarrow$ $m \in \{0,1,2\}$

Assume that $D_i \sim D$ s.t.

$$P(D \geq i) = 2^{-i}, \quad i = 0,1,2,\dots \qquad (4.2.4)$$

$$\Rightarrow \quad P(D = i) = 2^{-i} - 2^{-(i+1)}$$

$$E[(D-i)^+] = \sum_{K \geq 1} P((D-i)^+ \geq K) = \sum_{K \geq 1} P(D \geq i+K) \qquad (4.2.5)$$

$$\overset{(4.2.4)}{=} \sum_{K \geq 1} 2^{-(i+K)} = 2^{-i}\underbrace{\left(\sum_{K \geq 1} 2^{-K}\right)}_{=1} = 2^{-i}$$

Consider $m = 0$

$$\overset{(4.2.2)}{\Rightarrow} \quad P_{ij} = \begin{cases} P((3-D)^+ = j), & \text{if } i = 0 \\ P((i-D)^+ = j), & 0 < i \leq 3 \end{cases}$$

$$\Rightarrow \quad P_{00} = P((3-D)^+ = 0) = P(D \geq 3) \overset{(4.2.4)}{=} 2^{-3} = \frac{1}{8}$$

$$P_{01} = P((3-D)^+ = 1) = P(D = 2) = 2^{-2} - 2^{-3} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

$$P_{02} = P((3-D)^+ = 2) = P(D = 1) = 2^{-1} - 2^{-2} = \frac{1}{4}$$

$$P_{03} = P((3-D)^+ = 3) = P(D = 0) = 2^{-0} - 2^{-1} = \frac{1}{2}$$

$$P_{10} = P((1-D)^+ = 0) = P(D \geq 1) = 2^{-1} = \frac{1}{2}$$

$$P_{11} = P((1-D)^+ = 1) = P(D = 0) = 2^{-0} - 2^{-1} = \frac{1}{2}$$

$$\longrightarrow \quad P = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$\Pi \cdot P = \Pi$, $\sum_{i=0}^{3} \Pi_i = 1$ $\Longleftrightarrow$ $\Pi = \Pi(0) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ inv. distr.

$m = 1$:

$$P = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \quad \Pi = \Pi(1) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}\right)$$

$m = 2$:

$$P = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \quad \Pi = \Pi(2) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}\right)$$

$$\Rightarrow \quad \bar{f}(0) = \sum_{i=0}^{3} \Pi_i(0) f(i)$$

$$f(i) \overset{(4.2.5)}{=} E[(D-i)^+], \quad i = 1,\dots,3$$

$$\overset{(4.2.5)}{\Rightarrow} \quad f(i) = 2^{-i}, \quad i = 1,\dots,3 \quad \text{and} \quad f(0) = 2^{-3} = \frac{1}{8}$$

(82) $\Rightarrow \bar{f}(0) = \frac{1}{4}\left(\frac{1}{8} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{1}{4}$

Similarly $\bar{f}(1) = \frac{1}{6}$, $\bar{f}(2) = \frac{1}{8}$

and $\bar{g}(0) = \frac{1}{4}$, $\bar{g}(1) = \frac{1}{3}$, $\bar{g}(2) = \frac{1}{2}$

$\Rightarrow$ cost for $m=0$: $a\,\bar{g}(0) + \bar{f}(0) = \frac{a}{4} + \frac{1}{4}$

" " $m=1$: $a\,\bar{g}(1) + \bar{f}(1) = \frac{a}{3} + \frac{1}{6}$

" " $m=2$: $a\,\bar{g}(2) + \bar{f}(2) = \frac{a}{2} + \frac{1}{8}$

$\Rightarrow$ cost becomes for $m=0$ minimal, if $a \geq 1$

" " for $m=1$ minimal if $\frac{1}{4} \leq a \leq 1$

" " for $m=2$ minimal if $a \leq \frac{1}{4}$

Rem.: The above application is an example of ergodic control theory