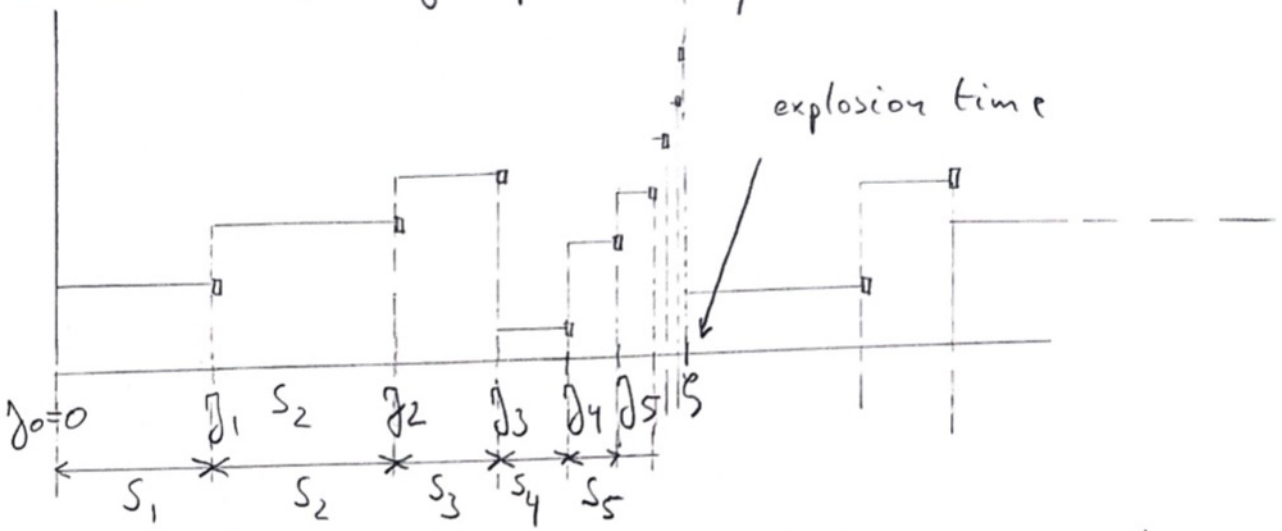
3. case: $X_s, s \geq 0$ jumps infinitely often on an interval $[0,t]$




$\longrightarrow$ $\tau_0, \tau_1, \tau_2, \ldots$ jump times of $X_s, s \geq 0$ and
$S_1, S_2, S_3, \ldots$ holding times

or more precisely: Def. stopping times $\tau_n$ by

$$\tau_{n+1} \stackrel{\text{def.}}{=} \inf\{t \geq \tau_n : X_t \neq X_{\tau_n}\}, \quad \tau_0 = 0 \qquad (5.1.2)$$

for $n \geq 0$, where $\inf \emptyset \stackrel{\text{def}}{=} \infty$

and random times by $S_n$ by

$$S_n = \begin{cases} \tau_n - \tau_{n-1}, & \text{if } \tau_{n-1} < \infty \\ \infty & \text{else} \end{cases} \qquad (5.1.3)$$

If $\tau_n < \infty$, but $\tau_{n+1} = \infty$ we define

$$X_\infty = X_{\tau_n} \qquad (5.1.4)$$

Further, the (first) explosion time $\zeta$ of $X_s, s \geq 0$ is defined by

$$\zeta = \sup_{n \geq 0} \tau_n = \sum_{n \geq 1} S_n \qquad (5.1.5)$$

In the sequel we call the proc.

$$Y_n \stackrel{\text{def.}}{=} X_{\tau_n}, \quad n \geq 0 \qquad (5.1.6)$$

the jump chain of $X_s, s \geq 0$, if $Y_n, n \geq 0$ is a discrete-time Markov chain

From now on we shall consider right-continuous proc. $X_t \in I \cup \{\infty\}$, which are minimal, that is

$$X_t = \infty, \qquad (5.1.7)$$

if $t \geq \zeta$

Rem.: A minimal proc. $X_s, s \geq 0$ may be reconstructed from its holding times $S_1, S_2, \ldots$ and its jump chain $Y_n, n \geq 0$

→ objective: Construction of minimal right-continuous Markov processes from discrete-time Markov chains by using exponentially distributed r.v's

Recall (see Ex. 2.11):

A r.v. $T \in [0, \infty]$ has exponential distribution of parameter $\lambda \geq 0$ ($T \sim \text{Exp}(\lambda)$ for short) if

$$P(T > t) = e^{-\lambda t}, \quad t \geq 0$$

If $\lambda > 0$, then the prob. density function $f_T$ of $T$ is given by

$$f_T(t) = \lambda e^{-\lambda t} 1_{[0,\infty)}(t)$$

Further

$$E[T] = \int_0^\infty P(T > t)\, dt = \frac{1}{\lambda}$$

→ important property:

Theorem 5.1.1 (Memoryless property)

A r.v. $T$ has an exponential distribution if and only if it has the memoryless property, that is

$$P(T > s+t \mid T > s) = P(T > t)$$

for all $s, t \geq 0$.

Proof: See Exercises.

## 5.2 Construction of continuous-time Markov chains

→ The construction of such a proc. is based on its jump chain (i.e. a discrete-time Markov chain) and its holding times

The basic data of a continuous-time Markov chain $X_{s,s\geq 0}$ are described by the Q-matrix (or generator matrix):

**Def. 5.2.1** (Q-matrix)
A matrix $Q=(q_{ij})_{i,j\in J}$ is called Q-matrix, if
(i) $0\leq -q_{ii}<\infty$ for all $i$,
(ii) $q_{ij}\geq 0$ for all $i\neq j$
(iii) $\sum_{j\in J} q_{ij}=0$ for all $i$

Rem: (i) A Q-matrix determines the evolution of a cont.-time Markov chain from its initial state (see later)
(ii) Notation: We use $q_i$ or $q(i)$ for $-q_{ii}$
(iii) Interpretation: $q_i$ rate of leaving $i$ and $q_{ij}$ the rate of jumping from $i$ to $j$

Ex. 5.2.2:
$$Q=\begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$$

→ diagram:




**Def. 5.2.3** (jump matrix $\Pi$ of Q)
The jump matrix $\Pi=(\pi_{ij})_{i,j\in J}$ of a Q-matrix $Q=(q_{ij})_{i,j\in J}$ is defined by

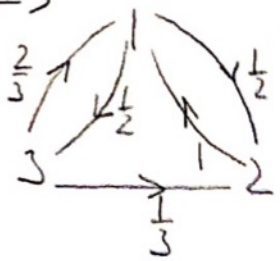
$$\pi_{ij}=\begin{cases} q_{ij}/q_i, & \text{if } j\neq i \text{ and } q_i\neq 0 \\ 0, & \text{if } j\neq i \text{ and } q_i=0 \end{cases}$$

$$\pi_{ii}=\begin{cases} 0, & \text{if } q_i\neq 0 \\ 1, & \text{if } q_i=0 \end{cases}$$

→ stochastic matrix! (of the jump chain of $X_{s,s\geq 0}$)

Ex. 5.2.4: Q of Ex. 5.2.2 ⟹

$$\Pi=\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 2/3 & 1/3 & 0 \end{pmatrix} \longrightarrow$$



**Def 5.2.5** (Markov chain $X_t, t\geq 0$ with generator $Q$)
A minimal (right-continuous proc. $X_t, t\geq 0$ on $\mathcal{I}$ is called *Markov chain with initial distr. $\lambda=(\lambda_i)_{i\in\mathcal{I}}$ and generator matrix $Q=(q_{ij})_{i,j\in\mathcal{I}}$* ($\text{Markov}(\lambda,Q)$ for short) if

(i) the jump chain $Y_n \overset{\text{def}}{=} X_{J_n}$ ($J_n$: jump time), $n\geq 0$ is $\text{Markov}(\lambda,\Pi)$ ($\Pi$ jump matrix)

(ii) for all $n\geq 1$, $i_0,\dots,i_{n-1}\in\mathcal{I}$, *conditional on* $Y_0=i_0,\dots,Y_{n-1}=i_{n-1}$ the holding times
$$S_1, S_2, \dots, S_n$$
are *indep.* and *exponentially distributed* with parameters $q(i_0), q(i_1), \dots, q(i_{n-1})$,
(i.e. $P(S_m>t \mid Y_0=i_0,\dots,Y_{n-1}=i_{n-1}) = e^{-q(i_{m-1})\cdot t}$, $m=1,\dots,n$)

→ *construction*: Choose (any) $Y_n, n\geq 0$ $\text{Markov}(\lambda,\Pi)$
Further let $T_n, n\geq 1$ be i.i.d. with $T_1\sim \text{Exp}(1)$ and independent of $Y_n, n\geq 0$
Define the *holding times* $S_n$ as
$$S_n = T_n / q(Y_{n-1}) \qquad (5.2.1)$$
and the *jump times* $J_n$ by
$$J_n = S_1 + S_2 + \dots + S_n \qquad (5.2.2)$$
Then
$$X_t \overset{\text{def}}{=} \begin{cases} Y_n, & \text{if } J_n \leq t < J_{n+1} \\ \infty & \text{else} \end{cases} \qquad (5.2.3)$$
is $\text{Markov}(\lambda,Q)$ (see also Exercises)

*Rem. 5.2.6*: By using the Markov property of $X_t, t\geq 0$ one shows (see p. 94 in Norris) that Def 5.2.5 is equivalent to the "*transition probability definition*", if $\mathcal{I}$ finite:
For all $n\geq 0$, $0\leq t_0<t_1<\dots<t_{n+1}$ and $i_0,\dots,i_{n+1}\in\mathcal{I}$:
$$P(X_{t_{n+1}}=i_{n+1} \mid X_{t_0}=i_0,\dots,X_{t_n}=i_n) = p_{i_n i_{n+1}}(t_{n+1}-t_n), \qquad (5.2.4)$$
Where
$$p_{ij}(t) = P(X_t=j \mid X_0=i), \quad i,j\in\mathcal{I}. \qquad (5.2.5)$$
Moreover $P(t) \overset{\text{def}}{=} (p_{ij}(t))_{i,j\in\mathcal{I}}$ is the solution of *Kolmogorov's forward equation*:

$$P'(t) = P(t) \cdot Q \quad , \quad P(0) = (\delta_{ij})_{i,j \in J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (5.2.6)$$

$$\Longleftrightarrow \quad \frac{d}{dt} P_{ij}(t) = \sum_{k \in J} P_{ik}(t)\, q_{kj} \quad , \quad P_{ij}(0) = \delta_{ij}$$

for all $i \in J$.

$\delta_{ij} = \begin{cases} 1 & , \text{if } i = k \\ 0 & \text{else} \end{cases}$ , $Q = (q_{ij})_{i,j \in J}$ Q-matrix

**Rem:** $P_{i_n i_{n+1}}(t_{n+1} - t_n) = P(X_{t_{n+1}} = i_{n+1} \mid X_{t_n} = i_n)$ in (5.2.4). So for all $n \geq 0$, $0 \leq t_0 < t_1 < \ldots < t_{n+1}$, $i_0, \ldots, i_{n+1} \in J$:

$$P(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \ldots, X_{t_n} = i_n) = P(X_{t_{n+1}} = i_{n+1} \mid X_{t_n} = i_n)$$

$\longrightarrow$ $X_t$, $t \geq 0$ process without memory

## 5.3 1. Example of Def. 5.2.5 : Poisson process

**Def 5.3.1** (Poisson process)
A proc. $X_t, t \geq 0$ in Def. 5.2.5 is called *Poisson process* if

(i) $J = \{0, 1, 2, \dots\}$

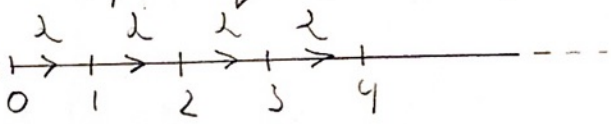
(ii) its holding times $S_1, S_2, \dots$ are i.i.d with $S_1 \sim Exp(\lambda)$

(iii) its jump chain is given by
$$Y_n = n \quad (\text{i.e. jumps of size } 1)$$

(iv) its generator matrix $Q$ is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & & 0 \\ & -\lambda & \lambda & \\ & & \ddots & \ddots \\ 0 & & & \ddots \end{pmatrix}$$

jump intensity




$\longrightarrow$ diagram:

*Rem*: SLLN: $\frac{1}{n} \underbrace{\sum_{i=1}^{n} S_i}_{\stackrel{def.}{=} J_n} \xrightarrow[n \to \infty]{} E[S_1] = \frac{1}{\lambda}$

$\Rightarrow J_n \xrightarrow[n \to \infty]{} \infty$ with prob. 1 $\Rightarrow$ no explosion (i.e. $\zeta = \infty$)

**Ex. 5.3.2** (Modeling of the total claim amount)
The modeling of the total claim amount (totalskadesum) in a portfolio of insurance policies (forsikringskontrakter) plays a central role in non-life insurance (skadeforsikring) and is a tool for premium calculation.

$\longrightarrow$ *Cramér-Lundberg model*: A popular model for the total claim amount $S(t)$ at time $t$:

$$S(t) = \sum_{i=1}^{N(t)} X_i = X_1 + \dots + X_{N(t)} \quad , t \geq 0 \quad , \quad (5.3.1)$$

where $X_i$ *claim size* (kravstørrelse) of a claim (e.g. fire loss) occuring at the *claim arrival time* $T_i$, $i \in \mathbb{N}$
$(0 \leq T_1 < T_2 < \dots)$

and $N(t) \in \mathbb{N}_0$ *claim numbers* given by

$$N(t) \stackrel{def}{=} \#\{i \geq 1 : T_i \leq t\}$$

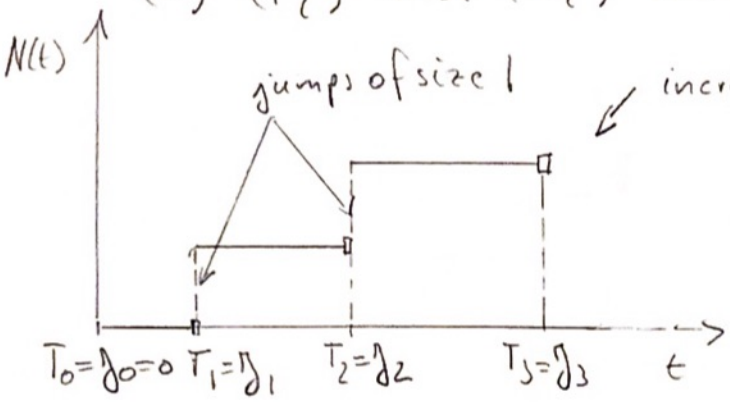
↑ number of

**Assumption:**

(i) $T_n = \gamma_n$ jump times of a Poisson process $N(t)$ with param. $\lambda > 0$

(ii) The claim sizes $X_i \geq 0$ are i.i.d

(iii) $(T_i)$ and $(X_i)$ independent ($\Rightarrow N(t)$ and $(X_i)$ indep.)




**Theorem 5.3.3** (Markov property of the Poisson proc.)

Let $X_t, t \geq 0$ be a Poisson proc. of rate $\lambda$.

Then, conditional on $X_s = i$ ($i \in \mathbb{N}, s \geq 0$), the proc. $\widetilde{X}_t \overset{\text{def}}{=} X_{s+t} - X_s, t \geq 0$ is a Poisson process of rate $\lambda$ and independent of $X_\tau, \tau \leq s$

**Proof:** Denote by $\widetilde{S}_1, \widetilde{S}_2, \ldots$ the holding times of $\widetilde{X}_t, t \geq 0$

We want to show the following:

Conditional on $X_s = i$ $\widetilde{S}_1, \widetilde{S}_2, \ldots$ are i.i.d with $\widetilde{S}_1 \sim \text{Exp}(\lambda)$ (5.3.2)

and indep. of $S_1, \ldots, S_i$

W.l.o.g. let us prove this for $\widetilde{S}_1, \widetilde{S}_2$:

$$\{X_s = i\} \overset{\text{jumps of size 1}}{\cap} \{\gamma_i \leq s < \underbrace{\gamma_{i+1}}_{= \gamma_i + S_{i+1}}\} = \{\gamma_i \leq s\} \cap \{S_{i+1} > s - \gamma_i\} \quad (*)$$

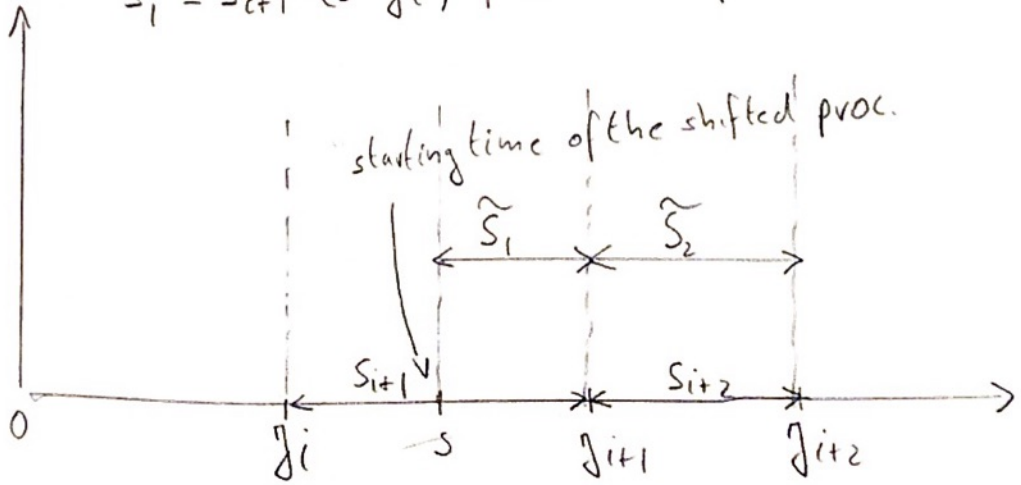
On this event

$$X_r = \sum_{j=1}^{i} \mathbb{1}_{\{\gamma_j \leq r\}} = \#\{1 \leq j \leq i : \gamma_j \leq r\}, \; r \leq s \quad (**)$$

(number of)

and

$$\widetilde{S}_1 = S_{i+1} - (s - \gamma_i), \; \widetilde{S}_n = S_{i+n} \text{ for } n \geq 2 \quad (***)$$



Then

$$P(\widetilde{S}_1 > x_1, \widetilde{S}_2 \le x_2 \mid S_1 \le y_1, \dots, S_i \le y_i \mid X_s = i)$$

$$\overset{(*),\,(***)}{=} P(S_{i+1} > x_1 + s - \eta_i \mid S_{i+2} \le x_2 \mid S_1 \le y_1, \dots, S_i \le y_i \mid \eta_i \le s \mid S_{i+1} > s - \eta_i)$$

$$= E\Big[ \mathbb{1}_{\{S_{i+1} > x_1 + s - \eta_i \mid S_{i+1} > s - \eta_i\}} \cdot \mathbb{1}_{\{S_{i+2} \le x_2\}} \cdot \mathbb{1}_{\{S_1 \le y_1, \dots, S_i \le y_i \mid \eta_i \le s\}} \Big]$$

"dobbelforventning" / $S_i$, $i \ge 1$ indep.

$$= E_{S_1, \dots, S_i}\Big[ \mathbb{1}_{\{S_1 \le y_1, \dots, S_i \le y_i \mid \eta_i \le s\}} \underbrace{E_{S_{i+2}}\big[\mathbb{1}_{\{S_{i+2} \le x_2\}}\big]}_{= P_{S_{i+2}}(S_{i+2} \le x_2)} \cdot \underbrace{\Big(E_{S_{i+1}}\big[\mathbb{1}_{\{S_{i+1} > x_1 + s - \eta_i \mid S_{i+1} > s - \eta_i\}}\big]\Big)}_{= P_{S_{i+1}}(S_{i+1} > x_1 + s - \eta_i \mid S_{i+1} > s - \eta_i)} \Big]$$

$$\cdot P_{S_{i+1}}(S_{i+1} > s - \eta_i)$$

memoryless property (Th. 5.1.1) $= P_{S_{i+1}}(S_{i+1} > x_1)$

$S_{i+1}, \widetilde{S}_2$ indep. $=$

$$P(S_{i+1} > x_1, S_{i+2} \le x_2) \cdot P(S_1 \le y_1, \dots, S_i \le y_i \mid \underbrace{\eta_i \le s}_{\overset{(*)}{=} X_s = i} \mid S_{i+1} > s - \eta_i) \qquad (+)$$

In the same way one shows that

$$P(S_{i+1} > x_1, S_{i+2} \le x_2) = P(\widetilde{S}_1 > x_1, \widetilde{S}_2 \le x_2 \mid X_s = i) \qquad (++)$$

$(+),(++)$ $\Rightarrow$

$$P(\widetilde{S}_1 \le x_1, \widetilde{S}_2 \le x_2, S_1 \le y_1, \dots, S_i \le y_i \mid X_s = i)$$

$$= \underbrace{P(\widetilde{S}_1 \le x_1, \widetilde{S}_2 \le x_2 \mid X_s = i)}_{= P(\widetilde{S}_1 \le x_1 \mid X_s = i)\, P(\widetilde{S}_2 \le x_2 \mid X_s = i)} P(S_1 \le y_1, \dots, S_i \le y_i \mid X_s = i) \qquad (+++)$$

$(++),(+++)$ $\Rightarrow$ claim (5.3.2)

Using (5.3.2) in connection with (**) the proof follows.

**Theorem 5.3.4**: Let $X_t, t \geq 0$ be an increasing, right-continuous proc. with values in $\mathbb{N} = \{0,1,2,\dots\}$ starting in zero. Let $0 < \lambda < \infty$. Then

(i) $X_t, t \geq 0$ is a Poisson proc. with param $\lambda > 0$ (Def 5.3.1)

$\Leftrightarrow$ (ii) *infinitesimal definition*: $X_t, t \geq 0$ has *independent increments*, that is the r.v.'s $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are indep. for all $0 \leq t_0 < t_1 < t_2 < \dots < t_n$ and for $h \searrow 0$, uniformly in $t$

$$P(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h), \quad P(X_{t+h} - X_t = 1) = \lambda h + o(h)$$

(convention: $f(t) = o(t) \iff \frac{f(t)}{t} \xrightarrow{t \to 0} 0$)

$\Leftrightarrow$ (iii) *transition probability definition*: $X_t, t \geq 0$ has *stationary increments*, that is

$$X_{t+h} - X_t \overset{d}{=} X_h$$

for all $t, h \geq 0$ the increm. of $X_t, t \geq 0$ are indep. and $X_t$ is Poisson distributed with param. $\lambda t$

**Proof**: (i) $\Rightarrow$ (ii): Because of the proof of Th. 5.3.3 (Markov prop. of $X_t, t \geq 0$, see (++)) we see that

$$P(X_{t+h} - X_t \leq x, X_t = i) = P(X_h \leq x) P(X_t = i) \qquad (*)$$

$\sum_i \Rightarrow$ on both sides $P(X_{t+h} - X_t \leq x) = P(X_h \leq x)$, first jump

$$\Rightarrow P(X_{t+h} - X_t \geq 1) = P(X_h \geq 1) = P(T_1 \leq h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$

($T_1 = S_1$)

and $P(X_{t+h} - X_t \geq 2) = P(X_h \geq 2) \overset{Y_h = n}{=} P(T_2 \leq h)$

($T_2 = S_1 + S_2$)

$$\leq P(S_1 \leq h, S_2 \leq h) \overset{S_1, S_2 \text{ indep and } \sim \text{Exp}(\lambda)}{=} (1 - e^{-\lambda h})^2 = o(h)$$

$$\Rightarrow P(X_{t+h} - X_t = 1) = 1 - P(X_{t+h} - X_t = 0 \text{ or } \geq 2) = \lambda h + o(h)$$

The independence of the increm of $X_t, t \geq 0$ follows from the Markov prop. (Th. 5.3.3) in connection with $(*)$

As for (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) see Norris.

## 5.4.2 Example of Def 5.2.5 : Birth processes

→ generalization of the Poisson process

**Def 5.4.1** (Birth process)

A proc. $X_t, t \geq 0$ in Def 5.2.5 is referred to as <u>birth process of rates $q_j, j \geq 0$</u> if

(i) $J = \{0, 1, 2, \dots\}$

(ii) conditional on $X_0 = i$, the holding times $S_1, S_2, \dots$ are indep. and exponentially distributed with param. $q_i, q_{i+1}, \dots$, respectively.

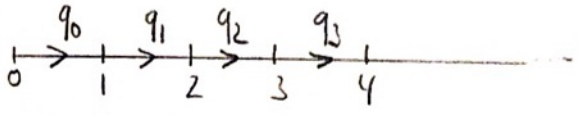
(iii) conditional on $X_0 = i$, its jump chain $Y_n$ is given by

$$Y_n = i + n, \quad n \geq 0 \quad \text{(jumps of size 1)}$$

(iv) its generator matrix $Q$ is given by

$$Q = \begin{pmatrix} -q_0 & q_0 & & \\ & -q_1 & q_1 & \\ & & -q_2 & q_2 \\ & \mathbf{0} & & \ddots \end{pmatrix}$$

→ diagram:




**Ex. 5.4.2** (Simple birth process)

Denote by $X_t$ the number of individuals of a population at time $t \geq 0$.

Suppose that $X_t = i$ and assume that each of the individuals independently gives birth after time

$$T_j \sim \text{Exp}(\lambda), \ \lambda > 0, \ j = 1, \dots, i$$

$\Rightarrow$ $S_1 \overset{\text{def}}{=} \min_{j=1}^{i} T_j$ time, when the 1. birth occurs

$$P(S_1 > t) = P(T_j > t, j = 1, \dots, i) \overset{T_j \sim \text{Exp}(\lambda) \text{ indep.}}{=} \prod_{j=1}^{i} e^{-\lambda t} = e^{-i\lambda t}$$

$\Rightarrow$ $S_1 \sim \text{Exp}(i\lambda)$

→ $i+1$ individuals

$\overset{\text{memoryless}}{\underset{\text{Prop. (Th. 5.1.1)}}{\Longrightarrow}}$ $S_2 \sim \text{Exp}((i+1)\lambda)$ (time betw. 1. and 2. birth)

→ $i+2$ individuals

⋮

→ <u>reasonable</u> to describe $X_t, t \geq 0$ by a birth process of rates $q_i = i\lambda, i \geq 0$

Rem: $E[X_t] = e^{\lambda t}$

In contrast to a Poisson proc. a birth proc. may explode:

**Theorem 5.4.3**: Suppose $X_t, t \geq 0$ is a birth proc. of rates $q_j, j \geq 0$ with $X_0 = 0$. Then explosion time

(i) If $\sum_{j \geq 0} \frac{1}{q_j} < \infty$, then $P(\zeta < \infty) = 1$.

(ii) If $\sum_{j \geq 0} \frac{1}{q_j} = \infty$, then $P(\zeta = \infty) = 1$

**Proof**: (i) $E[\sum_{i=1}^{n} S_i] = E[\sum_{i=1}^{n} S_i \mid X_0 = 0]$

$= \sum_{i=1}^{n} E[S_i \mid X_0 = 0] \overset{\text{Def 5.4.1, (ii)}}{=} \sum_{i=1}^{n} \frac{1}{q_{i-1}} \Rightarrow$

$E[\sum_{i \geq 1} S_i] < \infty \Rightarrow P(\zeta < \infty) = 1$

$\sum_{i \geq 1} S_i \overset{\text{def}}{=} \zeta$

(ii) see Norris.

Ex.: simple birth process of rates $q_i = i\lambda$

$\Rightarrow \sum_{i \geq 1} \frac{1}{q_i} = \infty \Rightarrow P(\zeta = \infty \mid X_0 = 1) = 1$

**Theorem 5.4.4** (Markov property of the birth process) Suppose that $X_t, t \geq 0$ is a birth proc. of rates $q_j, j \geq 0$. Then, conditional on $X_s = i$, the proc $\hat{X}_t \overset{\text{def}}{=} X_{s+t}, t \geq 0$ is a birth proc. of rates $q_j, j \geq 0$ starting from $i$ and indep. of $X_r, r \leq s$

**Proof**: Similar to the Poisson proc. case (Th 5.3.3)

**Theorem 5.4.5**: Assume $X_t, t \geq 0$ is an increasing right-continuous process with values in $\{0, 1, 2, \ldots\} \cup \infty$. Let $0 \leq q_j < \infty$ for all $j \geq 0$. Then

(i) $X_t, t \geq 0$ is a birth proc. of rates $q_j, j \geq 0$ (Def 5.4.1)

$\Leftrightarrow$

(ii) *infinitesimal definition*: Conditional on $X_t = i$, $X_{t+h}$ is indep. of $X_s, s \leq t$ for all $t, h \geq 0$. Further, as $h \downarrow 0$, uniformly in $t$

$$P(X_{t+h} = i \mid X_t = i) = 1 - q_i h + o(h)$$

and

$$P(X_{t+h} = i+1 \mid X_t = i) = q_i h + o(h)$$

$\Longleftrightarrow$

(iii) <u>transition probability definition</u>: For all $n \geq 0$, $0 \leq t_0 < t_1 < \dots < t_{n+1}$ and $i_0, \dots, i_{n+1} \in \mathbb{J} = \{0,1,2,\dots\}$:

$$P(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \dots, X_{t_n} = i_n) = p_{i_n i_{n+1}}(t_{n+1} - t_n),$$

where $p_{ij}(t) \overset{\text{def}}{=} P(X_t = j \mid X_0 = i)$, $i, j \geq 0$ uniquely solves <u>Kolmogorov's forward equation</u>

$$P'(t) = P(t) \cdot Q, \quad P(0) = (\delta_{ij}) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$\Longleftrightarrow$

$$\begin{cases} \frac{d}{dt} p_{i0}(t) = -p_{i0}(t)\, q_0, & p_{i0}(0) = \delta_{i0} \\ \frac{d}{dt} p_{ij}(t) = p_{i,j-1}(t)\, q_{j-1} - p_{ij}(t)\, q_j, & p_{ij}(0) = \delta_{ij}, \; j \geq 1 \end{cases}$$

$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{else} \end{cases}$

<u>Proof</u>: Similar to Th. 5.3.4