$$\{X_n = j\} = \dot{\bigcup}_{i_0,\dots,i_{n-1}\in J} \{X_0 = i_0, \dots, X_{n-1} = i_{n-1}, X_n = j\} \quad \text{(disjoint)}$$

Def. 2.2.6(ii) $\Rightarrow$

$$P(X_n = j) = \sum_{i_0,\dots,i_{n-1}\in J} P(X_0 = i_0, \dots, X_{n-1} = i_{n-1}, X_n = j)$$

Th. 3.1.3 $=$

$$\sum_{i_0,\dots,i_{n-1}\in J} \lambda_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} j} \overset{(3.1.5),\ (3.1.6)}{=} (\lambda p^n)_j$$

(ii) Apply (i) to the Markov chain $\tilde{X}_n := X_{m+n}$ (Th. 3.1.4) with $\lambda = \delta_i$ (see (3.1.3)) $\Rightarrow$ proof.

Ex. 3.1.6 (Virus mutation)

Denote by $1, \dots, N$ N diff. strains (stammer) of a virus! Consider a virus, which belongs to strain 1 in generation $n=0$, i.e. $X_0 = 1$

Assumption: The virus mutates from strain i in gen. n to another strain j in gen. $n+1$ with prob.

$$p_{ij} = \frac{\alpha}{N-1} \quad (i \neq j) \qquad (\alpha > 0)$$

$$(\Rightarrow \quad p_{ii} = 1 - (N-1)\cdot\frac{\alpha}{(N-1)} = 1-\alpha)$$

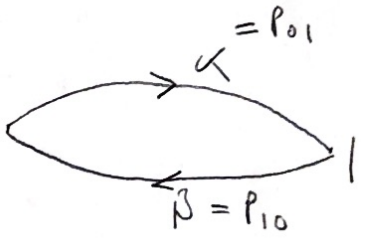
Denote by X_n with $X_0 = 1$ the strain of the virus in gen. n

$\longrightarrow$ possible model (for $(X_n)_{n\geq 0}$): Markov(δ_1, P) with P as above ($\overset{\text{Th. 3.1.5}}{=} p_{11}^{(n)}$)

$\longrightarrow$ Problem: $P(X_n = 1)$? (see Exerc. 3)

Ex. 3.1.7 ((Computation of $p_{00}^{(n)}$ for $J = \{0,1\}$))

$J = \{0,1\}$, $P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ $\longrightarrow$ diagram:



$p_{00}^{(n)}$ 2

$$p^{n+1} = p^n \cdot p \overset{\text{def}}{=} \underbrace{\begin{pmatrix} p_{00}^{(n)} & p_{01}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} \end{pmatrix}}_{= p^n} \cdot \underbrace{\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}}_{= P}$$

matrix multiplication $\Rightarrow$

$$p_{00}^{(n+1)} = p_{01}^{(n)} \cdot \beta + p_{00}^{(n)} (1-\alpha)$$

Note: $p_{00}^{(n)} + p_{01}^{(n)} \overset{\text{Th. 3.1.5(ii)}}{=} P_0(X_n = 0) + P_0(X_n = 1)$

$$= P_0(X_n \in \underbrace{\{0,1\}}_{=J}) = 1$$

$\Rightarrow$ recursion:

$$p_{00}^{(n+1)} = (1-\alpha-\beta)\, p_{00}^{(n)} + \beta, \quad p_{00}^{(0)} = 1$$

$\longrightarrow$ solution:

$$p_{00}^{(n)} = \begin{cases} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n, & \alpha+\beta > 0 \\ 1 \quad \text{for} \quad \alpha+\beta = 0 \end{cases}$$

See later

(25) **Ex**: Let S_n be the price of a stock at time n such that

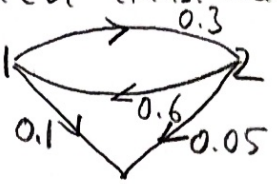
$$S_n = 0\$ \text{ or } 1\$ \text{ or } 2\$, \quad n = 0,1,2,\dots$$

and

$$P(S_0 = 1) = 1 \qquad (*)$$

Assumption: $(S_n)_{n\geq 0}$ Markov chain with trans. matr.

$$P = \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.6 & 0.35 \end{pmatrix} \longrightarrow$$



$\lambda = (\lambda_i)_{i\in J}$?

initial distr.: $\lambda_i \overset{\text{def.}}{=} P(S_0 = i) \overset{(*)}{=} \begin{cases} 1, & i = 1 \\ 0, & i = 0 \text{ or } 2 \end{cases}$

$\Longrightarrow \lambda = (0,1,0)$

$P(S_3 = 2)$?

Th. 3.1.5(c) $\longrightarrow$ formula: $P(S_n = j) = (\lambda P^n)_j$

$n=3, j=2$ $\Longrightarrow P(S_3 = 2) = (\lambda P^3)_2$

$(\lambda P^3)_2$?

Recall: λM for $M = (m_{ij})_{i,j\in J} = \begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & - & - \\ m_{20} & - & m_{22} \end{pmatrix}$ is def. as

$$\lambda M = ((\lambda M)_0, (\lambda M)_1, (\lambda M)_2)$$

with $(\lambda M)_j \overset{(3.1.5)}{=} \sum_{i\in J} \lambda_i m_{ij} \overset{J=\{0,1,2\}}{=} \sum_{i=0}^{2} \lambda_i m_{ij}, \ j = 0,1,2 \quad (**)$

Choose $M = P^3 = P\cdot P\cdot P \overset{\text{def.}}{=} \begin{pmatrix} p_{00}^{(3)} & p_{01}^{(3)} & p_{02}^{(3)} \\ p_{10}^{(3)} & - & - \\ p_{20}^{(3)} & - & p_{22}^{(3)} \end{pmatrix}$

P^3 ?

$$P^2 = P\cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0.175 & 0.54 & 0.285 \\ 0.1275 & 0.57 & 0.3025 \end{pmatrix}$$

$$\Longrightarrow P^3 = P^2\cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0.24325 & 0.495 & 0.26175 \\ 0.199625 & 0.5235 & 0.276875 \end{pmatrix}$$

($p_{12}^{(3)}$ = 0.26175)

$\lambda = (0,1,0)$, $(**)$, $M = P^3$ $\Longrightarrow (\lambda M)_j = 1\cdot m_{1j} = m_{1j}$

$\Longrightarrow (\lambda P^3)_j = p_{1j}^{(3)}$

$j = 2$ $\Longrightarrow (\lambda P^3)_2 = p_{12}^{(3)} = 0.26175$

3.2 Hitting times and absorption probabilities

Def. 3.2.1 (Class structure)
Let $(X_n)_{n\geq 0}$ be a Markov chain

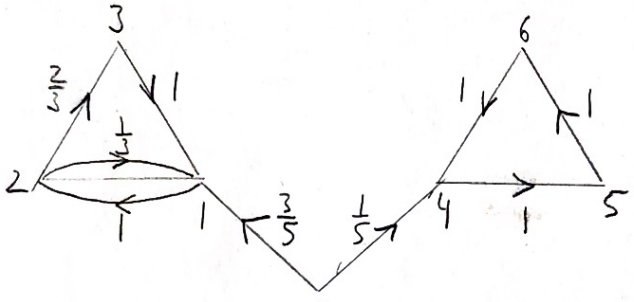
(i) We shall say that i leads j (or $i \to j$), if
$$P_i(X_n = j \text{ for some } n \geq 0) > 0 \qquad (P_i(A) \overset{\text{def}}{=} P(A \mid X_0 = i))$$
$\to$ starting in i the chain will reach j at some time n with pos. probability

(ii) i communicates with j (or $i \leftrightarrow j$) if $i \to j$ and $j \to i$

(iii) $C \subseteq I$ is a communicating class, if there ex. $j \in I$:
$$C = \{i \in I : i \leftrightarrow j\}$$

(iv) C in (iii) is called closed, if
$$i \in C \text{ and } i \to j \implies j \in C$$
$\to$ once X_n reaches C it cannot escape from it, since: $i \to j$ and $j \to k \implies i \to k$ (Exc.)

Ex.:



$I = \{0, \dots, 6\}$

(iii), (iv) $\to$ $\{0\}$, $\{1,2,3\}$, $\{4,5,6\}$ communicat. classes
$\{4,5,6\}$: closed classes

Def. 3.2.2 (Hitting time)
Let $(X_n)_{n\geq 0}$ be a Markov chain with trans. matrix P
The hitting time of a subset $A \subseteq I$ is the r.v.
$$H^A : \Omega \to \{0,1,2,\dots\} \cup \{\infty\}$$
def. by
$$H^A(\omega) := \inf\{n \geq 0 : X_n(\omega) \in A\}$$
$(\inf \emptyset := \infty)$
$$h_i^A := P_i(H^A < \infty) \quad (\overset{\text{def}}{=} P(H^A < \infty \mid X_0 = i)) \quad (3.2.1)$$
$\to$ the prob. that the chain starting in i ever hits A
If A is closed, h_i^A is called absorption probability
$$k_i^A := E_i[H^A] \quad (\overset{\text{def}}{=} E[H^A \mid X_0 = i]) \quad (3.2.2)$$
$\to$ mean time that the chain starting in i ever reaches A

Rem.: We shall write
$$h_i^A = P_i(\text{hit } A) \text{ and } k_i^A = E_i[\text{time to hit } A]$$

(27) **Ex. 3.2.3** (Gambler's ruin)

Denote by $X_n \in \mathbb{J} := \{0,1,2,\dots\}$ the wealth of a gambler in a casino at time n (in €)

Assumption: $(X_n)_{n\geq 0}$ modelled by a Markov chain with trans. prob.

$$p_{00} = 1$$
$$p_{i,i-1} = q \;,\; p_{i,i+1} = p \;,\; i \geq 1$$

where $0 < p := 1-q < 1$

→ diagram:



→ $h_i := P_i(\text{hit } 0)$ prob. that the gambler with initial wealth i gets ruined (since $A=\{0\}$ is closed)

$A=\{0\}$ →

Th. 3.2.4 (Computation of hitting prob.)

The hitting prob. h_i^A, $i \in \mathbb{J}$ are given by the minimal (non-negative solution of the linear equations)

$$\begin{cases} h_i^A = 1 & i \in A \\ h_i^A = \sum_{j\in\mathbb{J}} p_{ij} h_j^A & i \notin A \end{cases} \qquad (3.2.3)$$

Rem: minimal means that if x_i, $i \in \mathbb{J}$ is another solution with $x_i \geq 0$, $i \in \mathbb{J}$ then $x_i \geq h_i^A$ for all i

Proof (i) h_i^A, $i \in \mathbb{J}$ solves (3.2.3):

1. case: $X_0 = i \in A \implies H^A = 0 \implies h_i^A = 1$

2. case: $X_0 = i \notin A \implies H^A \geq 1$

$$\implies P_i(H^A < \infty \mid X_1 = j) \overset{\text{def. of cond. prob.}}{=} \frac{P(H^A < \infty, X_0 = i, X_1 = j)}{P(X_0 = i, X_1 = j)}$$

$$\overset{\text{def. of cond. prob.}}{=} P(\underbrace{H^A < \infty, X_0 = i}_{= \{X_n \in A \text{ for some } n \geq 1\} \cap \{X_0 = i\} =: B} \mid X_1 = j) \cdot \frac{1}{P(X_0 = i \mid X_1 = j)} \qquad (*)$$

$B, \{X_0 = i\}$ indep. conditional on $\{X_1 = j\}$ (see Th. 3.1.4)

$$(*) = P(B \mid X_1 = j) \cdot P(\{X_0 = i\} \mid X_1 = j) \cdot \frac{1}{P(X_0 = i \mid X_1 = j)}$$

$$= P(X_n \in A \text{ for some } n \geq 1 \mid X_1 = j)$$

$$\overset{\text{Exerc. 4}}{=} P(X_n \in A \text{ for some } n \geq 0 \mid X_0 = j)$$

$$\overset{\text{def.}}{=} P_j(H^A < \infty) \overset{\text{def. } h_j^A}{=} h_j^A \qquad (**)$$

$$\implies h_i^A \overset{\text{def.}}{=} P_i(H^A < \infty) = P_i\Big(\bigcup_{j\in\mathbb{J}} \{H^A < \infty, X_1 = j\}\Big)$$

$$\overset{\text{Def. 2.2.(c)}}{=} \sum_{j\in\mathbb{J}} P_i(H^A < \infty, X_1 = j)$$

$$\overset{\text{def. of cond. prob.}}{=} \sum_{j\in\mathbb{J}} P_i(H^A < \infty \mid X_1 = j) \cdot \underbrace{P_i(X_1 = j)}_{= p_{ij}}$$

$$\overset{(**)}{=} \sum_{j\in\mathbb{J}} h_j^A \cdot p_{ij} \implies (i)$$

(27)+ (ii) $\underline{h_i^A, i \in J \text{ min. solution}}$: Let $x_i, i \in J$ be a solution
ea (3.2.3) $\Rightarrow$ $h_i^A = x_i = 1$ for $i \in A$

Let $i \notin A$

(3.2.3) $\Rightarrow$ $$x_i = \sum_{j \in J} p_{ij} x_j = \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} x_j \qquad (+)$$

$$\Rightarrow x_j = \sum_{k \in A} p_{jk} + \sum_{k \notin A} p_{jk} \cdot x_k$$

(+) $\Rightarrow$ $$x_i = \underbrace{\sum_{j \in A} p_{ij}}_{= P_i(X_1 \in A)} + \underbrace{\sum_{j \notin A} p_{ij} \left(\sum_{k \in A} p_{jk} + \sum_{k \notin A} p_{jk} \cdot x_k \right)}_{\overset{\text{Th. 3.1.3}}{=} P_i(X_1 \notin A, X_2 \in A)}$$

$$= P_i(X_1 \in A) + P_i(X_1 \notin A, X_2 \in A) + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} \cdot x_k$$

Repeating the above procedure gives

$$x_i = \underbrace{P_i(X_1 \in A)}_{= H^A = 1} + \ldots + \underbrace{P_i(X_1 \notin A, \ldots, X_{n-1} \notin A, X_n \in A)}_{= P_i(H^A \leq n)} + \underbrace{\sum_{j_1 \notin A} \cdots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n} \cdot x_{j_n}}_{\geq 0}$$

$\Rightarrow$ $x_i \geq P_i(H^A \leq n)$ for all n

$\overset{n \to \infty}{\Rightarrow}$ $x_i \geq P_i(H^A < \infty) \overset{\text{def.}}{=} h_i^A$, $i \in J$

$\Rightarrow$ (ii) $\Rightarrow$ proof

See Th. 1.3.2 in Norris

Solution to Ex. 3.2.3 (Gambler's ruin):

By Th. 3.2.4 $h_i^A = h_i = P_i(\text{hit } 0)$ is the minimal non-neg. solution of

$$\begin{cases} h_i^A = 1, & i \in A = \{0\} \\ h_i^A = \sum_{j \in \mathbb{N}} p_{ij} h_j^A, & i \notin A = \{0\} \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} h_0 = 1 \\ h_i = p h_{i+1} + q h_{i-1}, & i = 1, 2, \dots \end{cases} \qquad (3.2.4)$$

1. $p \neq q$: One shows that (3.2.4) has a general solution of the form $h_i = A + B\left(\frac{q}{p}\right)^i$

See Sect. 1.11 in Norris.

1.1 $p < q$ (usual in casinos): $\left(\frac{q}{p}\right)^i \xrightarrow{i \to \infty} \infty$

$0 \le h_i \le 1 \Longrightarrow B = 0 \Longrightarrow h_i = A$ for all i

$i = 0 \Longrightarrow 1 = h_0 = A \Longrightarrow h_i = 1$ for all i

$\Longrightarrow$ ruin sure

1.2 $p > q$: $i = 0 \Rightarrow 1 = h_0 = A + B \cdot \left(\frac{q}{p}\right)^0 = A + B$

$\Longrightarrow B = 1 - A \xrightarrow{i \to \infty} 0$

$\Longrightarrow h_i = \left(\frac{q}{p}\right)^i + A\left(1 - \left(\frac{q}{p}\right)^i\right)$

$h_i \ge 0, i \in \mathbb{N} \Longrightarrow A \ge 0.$

$A = 0 \Longrightarrow h_i = \left(\frac{q}{p}\right)^i$ minimal non-neg. solution.

$\Longrightarrow$ ruin prob. small (if the initial wealth i is large (e.g. $q = 0.4$, $i = 10$ $\$$ $\Longrightarrow h_{10} \approx 1.7\%$)

2. $p = q$: (3.2.4) has the general solution

$0 \le h_i \le 1 \Longrightarrow h_i = A + B \cdot i, \quad i \in \mathbb{N}$

$B = 0 \Longrightarrow h_i = A, i \in \mathbb{N} \xRightarrow{i=0} A = 1$

$\Longrightarrow h_i = 1, i \in \mathbb{N}$

$\Longrightarrow$ ruin sure (even in a fair casino)