generalization →

Ex. 3.2.5 (Birth-and-death chain)

Consider a Markov chain $(X_n)_{n\geq 0}$ with trans. prob.

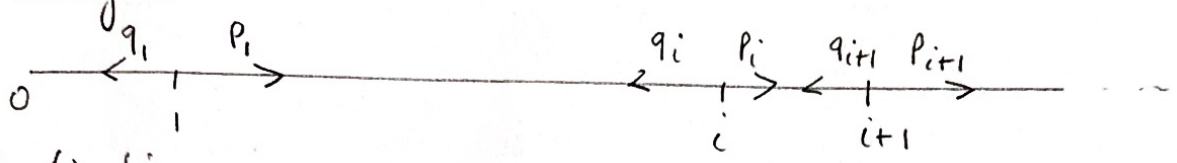
$$p_{00} = 1$$

$$p_{i,i-1} = q_i \,,\; p_{i,i+1} = p_i \,,\; i \geq 1 \qquad (*)$$

(all other trans. prob. are zero!)

$0 < p_i := 1 - q_i < 1 \,,\; i \geq 0$

→ diagram:



→ Application:

(i) Gambler's ruin with p and q (in Ex. 3.2.3) depending on the wealth i

(ii) X_n describes the size of a population in generation n. The population can either increase or decrease by (the unit) 1 (e.g. 10000 individuals) from one gen. to the other with prob. p_i and q_i (at size i), respectively.

If the pop reaches 0, it will be extinct. (closed class)

→ $h_i = P_i(\text{hit } 0)$ ($\stackrel{\text{def.}}{=} P(H^A < \infty \mid X_0 = i)$ with $A = \{0\}$) (absorption prob.)

probability that the population with initial size i ever disappears

By Th. 3.2.4 $h_i^A = h_i = P_i(\text{hit } 0)$ is the minimal non-neg. solution of

$$\begin{cases} h_i^A = 1 \,, & i \in A = \{0\} \\ h_i^A = \sum_{j \in J} p_{ij} h_j^A \,, & i \notin A = \{0\} \end{cases}$$

$\overset{(*)}{\Longleftrightarrow}$ $J = \{0,1,2,\dots\}$

$$\begin{cases} h_0 = 1 \\ h_i = p_i h_{i+1} + q_i h_{i-1} \,, & i = 1,2,\dots \end{cases} \qquad (3.2.5)$$

Let us derive a gen. solution to (3.2.5):

Substitute $u_i := h_{i-1} - h_i \,,\; i \geq 1$ in (3.2.5)

(30)

(3.2.5) $h_i = 1 \cdot h_i = \overbrace{(p_i+q_i)}^{=1} \cdot h_i$

$\Rightarrow \quad (p_i+q_i)\cdot h_i = p_i h_{i+1} + q_i h_{i-1}$

$\Leftrightarrow \quad p_i \underbrace{(h_i - h_{i+1})}_{\overset{\text{def}}{=} u_{i+1}} = q_i \underbrace{(h_{i-1} - h_i)}_{\overset{\text{def}}{=} u_i}$

$\Rightarrow \quad p_i u_{i+1} = q_i u_i$

$\Rightarrow \quad u_{i+1} = \left(\frac{q_i}{p_i}\right) u_i = \left(\frac{q_i}{p_i}\right)\cdot\left(\frac{q_{i-1}}{p_{i-1}} u_{i-1}\right) = \left(\frac{q_i q_{i-1}}{p_i p_{i-1}}\right) u_{i-1}$ (u_i = the bracketed factor)

$= \ldots = \underbrace{\left(\frac{q_i q_{i-1} \cdot \ldots \cdot q_1}{p_i \cdot p_{i-1} \cdot \ldots \cdot p_1}\right)}_{=: \gamma_i} \cdot u_1 = \gamma_i \cdot u_1, \; i \geq 0 \; (+)$

Note:

$u_1 + u_2 + \ldots + u_i \overset{\text{def}}{=} (h_0 - h_1) + (h_1 - h_2) + \ldots + (h_{i-1} - h_i)$

$\overset{\text{telescope sum}}{=} h_0 - h_i \overset{=1}{=} 1 - h_i$

On the other hand:

$u_1 + u_2 + \ldots + u_i \overset{(+)}{=} \gamma_0 \cdot u_1 + \gamma_1 \cdot u_1 + \ldots + \gamma_{i-1} u_1$

$= u_1 (\gamma_0 + \gamma_1 + \ldots + \gamma_{i-1})$

$\Rightarrow \quad 1 - h_i = \underbrace{u_1}_{=: A} (\gamma_0 + \gamma_1 + \ldots + \gamma_{i-1})$

$\Rightarrow \quad h_i = 1 - A(\gamma_0 + \ldots + \gamma_{i-1})$ ← gen. solution to (3.2.5)! (++)

where $A = u_1$, $\gamma_0 = 1$ → remains to find A:

1. case: $\sum_{i \geq 0} \gamma_i = \infty$ (e.g. $q_i = \frac{2}{3}$, $p_i = \frac{1}{3}$):

$0 \leq h_i \leq 1 \Rightarrow$ (++) $A = 0$ (otherwise $h_i \xrightarrow[i\to\infty]{} \pm\infty$) $\Rightarrow h_i = 1$ for all i

$\Rightarrow$ sure extinction of the pop.

2. case: $\sum_{i \geq 0} \gamma_i < \infty$ (e.g. $q_i = \frac{1}{3}$, $p_i = \frac{2}{3}$):

(++) $\Rightarrow$ $0 \leq h_i \leq 1 \quad 1 - A(\gamma_0 + \ldots + \gamma_{i-1}) \geq 0$ for all i

$\Rightarrow \quad \frac{1}{\gamma_0 + \ldots + \gamma_{i-1}} \geq A \geq 0$ for all i

$\Rightarrow \quad \frac{1}{\sum_{i \geq 0} \gamma_i} \geq A \geq 0 \qquad (+++)$

$\Rightarrow$ for all such A $h_i = 1 - A(\gamma_0 + \ldots + \gamma_{i-1})$ is a solution of (3.2.5)

$\Rightarrow A = \frac{1}{\sum_{i \geq 0} \gamma_i}$ gives the min. solution

$\Rightarrow \quad h_i = 1 - \frac{1}{\sum_{i \geq 0} \gamma_i} (\gamma_0 + \ldots + \gamma_{i-1})$ min. solut. (3.2.6)

→ extinction prob. of a pop. with initial size i

(31) e.g. $q_i = \frac{1}{3}$, $p_i = \frac{2}{3}$ $\Rightarrow \gamma_i = \left(\frac{1}{2}\right)^i$ $(\gamma_0 = 1)$

$\overset{\text{geom. sum}}{=\!=\!\Rightarrow}$ $\sum_{i\geq 0} \gamma_i = \frac{1}{1-\frac{1}{2}} = 2$, $\sum_{j=0}^{i-1} \gamma_j = \frac{1-\left(\frac{1}{2}\right)^i}{1-\frac{1}{2}}$ $\overset{(3.2.6)}{\Rightarrow} h_i = \left(\frac{1}{2}\right)^i$ (e.g. $i=5 \to h_5 \approx 3.1\%$)

$K_i^A \overset{\text{def.}}{=} E[H^A \mid K_0 = i]$?

→ **Th. 3.2.6** (Computation of the mean hitting time K_i^A)

The mean hitting times K_i^A, $i \in \mathbb{N}$ are given by the min. non-neg. solution of the linear equations

$$\begin{cases} K_i^A = 0, & i \in A \\ K_i^A = 1 + \sum_{j \notin A} p_{ij} K_j^A, & i \notin A \end{cases} \qquad (3.2.7)$$

Proof: similar to Th 3.2.4

Ex. 3.2.7 Consider a car insurance with the risk categories

1 ≙ "low risk", 2 ≙ "intermediate risk", 3 ≙ "high risk"

Suppose that in any given year the insured has

(i) no accidents with prob. 0.6
(ii) one accident with prob. 0.2
(iii) two accidents with prob. 0.1
(iv) more than two accidents with prob. 0.1

Further

no accidents ⟶ customer is moved down one risk category
one accident ⟶ " stays in the category
two accidents ⟶ " moves up one category
more than two accidents ⟶ " moves to high risk

Denote by X_n the risk category of the insured in year $n \geq 0$

⟶ $X_{n}, n \geq 0$ can be modelled as a Markov chain with state space $\mathbb{I} = \{1,2,3\}$ and transition prob.

$p_{11} = 0.6 + 0.2 = 0.8$
$p_{12} = 0.1$
$p_{13} = 0.1$
$p_{21} = 0.6$
$p_{22} = 0.2$
$p_{23} = 0.1 + 0.1 = 0.2$
$p_{31} = 0$
$p_{32} = 0.6$
$p_{33} = 0.2 + 0.1 + 0.1 = 0.4$



⟶ $X_{n}, n \geq 0$ Markov chain with $X_0 = 1$ and $P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0.6 & 0.4 \end{pmatrix}$

What is the mean time that the customer starting in the lowest risk category reaches the highest?

Or more mathematically

k_i^A ?

for $i = 1$ and $A = \{3\}$

Recall: $k_i^A \overset{\text{def}}{=} \mathbb{E}[H^A \mid X_0 = i]$

⟶ mean time that the chain starting in i ever hits A

$H^A := \inf\{n \geq 0 : X_n \in A\}$ hitting time

(33) By Th. 3.2.6 $K_i^A, i \in \mathbb{N}$ is the min. non-neg. solution of the linear equations

$$\begin{cases} K_i^A = 0, & i \in A \\ K_i^A = 1 + \sum_{j \notin A} p_{ij} K_j^A, & i \notin A \end{cases} \quad (*)$$

$A = \{3\}$

$$\Longleftrightarrow \begin{cases} K_i^A = 0, & i \in \{3\} \\ K_i^A = 1 + \sum_{j \in \{1,2\}} p_{ij} K_j^A, & i \in \{1,2\} \end{cases}$$

$$\Longleftrightarrow \begin{cases} K_3^A = 0 \\ K_1^A = 1 + \overset{=0.8}{p_{11}} K_1^A + \overset{=0.1}{p_{12}} K_2^A \\ K_2^A = 1 + \overset{0.6}{p_{21}} K_1^A + \overset{=0.2}{p_{22}} K_2^A \end{cases}$$

$$\Rightarrow \quad 0.8 K_2^A = 1 + 0.6 K_1^A \Rightarrow K_2^A = \frac{5}{4} + \frac{3}{4} K_1^A$$

$$\Rightarrow \quad K_1^A = 1 + 0.8 K_1^A + 0.1 \left(\frac{5}{4} + \frac{3}{4} K_1^A \right)$$

$$\Rightarrow \quad 0.2 K_1^A = 1 + \frac{5}{40} + \frac{3}{40} K_1^A$$

$$\Rightarrow \quad \frac{5}{40} K_1^A = \frac{45}{40} \Rightarrow K_1^A = 9$$

$$\Rightarrow K_2^A = 8$$

$\Rightarrow$ (*) has the unique solution $K_3^A = 0, K_1^A = 9, K_2^A = 8$

$\Rightarrow$ min. non-neg. solution

$\Rightarrow K_1^A = 9$ years

3.3 Strong Markov property

Consider the price of a stock S_n at time n such that

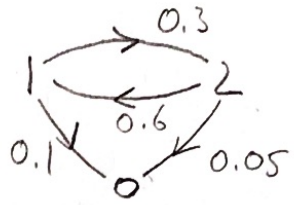
$$S_n = 0\$ \text{ or } 1\$ \text{ or } 2\$ \,, \quad n = 0,1,2,\dots$$

and

$$P(S_0 = 1) = 0.7 \,, \quad P(S_0 = 2) = 0.3$$

with trans. matrix P given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.6 & 0.35 \end{pmatrix}$$



Suppose there is a trader who buys the stock at a cheap price, i.e. when $S_n = 1\$$, $n \geq 0$

→ problems:

What is the probability that the stock price reaches its maximal value ($2\$$) 3 months after its purchase ($n=3$)?

Or more mathematically:

$$P(\tilde{S}_n = 2\$ \mid H^A < \infty) \,?$$

where $\tilde{S}_n := S_{H^A + n}$, $n \geq 0$ (random time)

for the hitting time

$$H^A = \inf\{n \geq 0 : S_n \in A\}$$

with $A = \{1\$\}$ (hitting time of $\tilde{S}_n$ with $\tilde{A} = \{2\$\}$)

Or what is the expected time after the purchase that S_n hits $2\$$, that is $E[\tilde{H}^{\tilde{A}} \mid H^A < \infty]$?

→ solution to these problems can be obtained by exploiting the Markov property of the random time shifted process $\tilde{S}_n$

→ strong Markov property (see later)

which holds more generally for random times given by stopping times

Def. 3.3.1 (stopping time)

Let $X_n, n \geq 0$ be a process. Then a random variable T with values in $\{0,1,2,\dots\} \cup \{\infty\}$

is called <u>stopping time</u> with respect to $X_n, n\geq 0$, if the event $\{T=m\}$ is determined by $X_0,\dots,X_m$ for all $m\geq 0$, that is

$$\{T=m\} = \bigcup_{k\geq 1} A_k,$$

where A_k is of the form $\{X_0=i_0,\dots,X_m=i_m\}$ (see (3.1.2))

<u>Ex. 3.3.2</u>: (i) <u>First passage time</u> (første passagetid):

$T_j := \inf\{n\geq 1 : X_n = j\}$ stopping time, since:

$\{T_j = m\} = \{X_1\neq j, X_2\neq j,\dots,X_{m-1}\neq j, X_m = j\}$ ← event determined by $X_0,\dots,X_m$

(ii) hitting time (første træffetid):

$H^A = \inf\{n\geq 0 : X_n\in A\}$ stopping time, since:

$\{H^A=m\} = \{X_0\notin A,\dots,X_{m-1}\notin A, X_m\in A\}$

(iii) <u>Last exit time</u>:

$L^A := \sup\{n\geq 0 : X_n\in A\}$ is in general <u>not</u> a stopping time, since:

$\{L^A=m\} = \{X_m\in A, X_{m+1}\notin A, X_{m+2}\notin A,\dots\}$ ← in general determ. by $X_m, X_{m+1}, X_{m+2},\dots$

Let T be a stopping time. In what follows we shall say that an event B is <u>determined by</u> $X_0, X_1,\dots, X_T$ if the event

$$B\cap\{T=m\} \qquad (3.3.1)$$

is determined by $X_0, X_1,\dots, X_m$

<u>Th. 3.3.3</u> (Strong Markov property)

Let $X_n, n\geq 0$ be Markov(λ, P). Assume that T is a stopping time w.r.t $X_n, n\geq 0$. Then the <u>random time-shifted</u> proc. $\widetilde{X}_n := X_{T+n}, n\geq 0$ is Markov(δ_i, P) with respect to the (conditional) probability

$$\widetilde{P}(A) := P(A \mid T<\infty, X_T = i)$$

Further the Markov chain $\widetilde{X}_n, n\geq 0$ is independent of $X_0, X_1,\dots, X_T$ given $T<\infty, X_T = i$, that is

(56)

$$P(\{\widetilde{X}_0 = X_T, \widetilde{X}_1 = X_{T+1}, \widetilde{X}_n = X_{T+n} = j_0, \widetilde{X}_1 = j_1, \dots, \widetilde{X}_n = j_n\} \cap B \mid T<\infty, X_T = i)$$

$$= P(\{\widetilde{X}_0 = j_0, \widetilde{X}_1 = j_1, \dots, \widetilde{X}_n = j_n\} \mid T<\infty, X_T = i)$$

$$\cdot P(B \mid T<\infty, X_T = i) \qquad (3.3.2)$$

for all $j_0, \dots, j_n \in J$, $n \geq 0$ and B determined by $X_0, \dots, X_T$