(56)

$$P(\{\widetilde{X}_0 = \overset{=X_T}{j_0}, \widetilde{X}_1 = \overset{=X_{T+1}}{j_1}, \dots, \widetilde{X}_n = \overset{=X_{T+n}}{j_n}\} \cap B \mid T<\infty, X_T = i)$$
$$= P(\{\widetilde{X}_0 = j_0, \widetilde{X}_1 = j_1, \dots, \widetilde{X}_n = j_n\} \mid T<\infty, X_T = i)$$
$$\cdot P(B \mid T<\infty, X_T = i) \qquad (3.3.2)$$

for all $j_0, \dots, j_n \in J$, $n \geq 0$ and B determined by $X_0, \dots, X_T$

Proof: We check the def. of a Markov chain by using the Markov property of $X_n, n \geq 0$ (Th. 3.1.4)

1. $\widetilde{P}(\widetilde{X}_0 = j) \overset{\text{def.}}{=} P(X_T = j \mid T<\infty, X_T = i)$
$$= \begin{cases} 1, & i = j \\ 0 & \text{else} \end{cases}$$
$\Rightarrow$ initial distr. $\lambda = \delta_i = (0, \dots, 0, \underset{\uparrow}{1}, 0, \dots)$ (i-th position)

$p_{i_n i_{n+1}} \overset{!}{=}$

2. $\widetilde{P}(\widetilde{X}_{n+1} = i_{n+1} \mid \widetilde{X}_0 = i_0, \dots, \widetilde{X}_n = i_n)$
$\overset{\text{def. of cond. prob.}}{=} \dfrac{\widetilde{P}(\widetilde{X}_{n+1} = i_{n+1}, \widetilde{X}_0 = i_0, \dots, \widetilde{X}_n = i_n)}{\widetilde{P}(\widetilde{X}_0 = i_0, \dots, \widetilde{X}_n = i_n)}$
$\overset{\text{def. of } \widetilde{P}}{=} \dfrac{P(\widetilde{X}_{n+1} = i_{n+1}, \widetilde{X}_0 = i_0, \dots, \widetilde{X}_n = i_n \mid T<\infty, X_T = i)}{P(\widetilde{X}_0 = i_0, \dots, \widetilde{X}_n = i_n \mid T<\infty, X_T = i)}$
$\overset{\text{def. of cond. prob. and } X_{n+1}, n \geq 0}{=} \dfrac{P(X_{T+n+1} = i_{n+1}, X_T = i_0, \dots, X_{T+n} = i_n, T<\infty, X_T = i)}{P(X_T = i_0, \dots, X_{T+n} = i_n, T<\infty, X_T = i)} =: Z$ (*)

disjoint

Note: $\{T<\infty\} = \dot{\bigcup}_{m \geq 0} \{T = m\}$ (**)

$\Rightarrow Z = P(\dot{\bigcup}_{m \geq 0} \{X_{T+n+1} = i_{n+1}, X_T = i_0, \dots, X_{T+n} = i_n, T = m, X_T = i\})$
$\overset{\text{def. of prob.}}{=} \sum_{m \geq 0} P(X_{T+n+1} = i_{n+1}, X_T = i_0, \dots, X_{T+n} = i_n, T = m, X_T = i)$
$\overset{T=m}{=} \sum_{m \geq 0} P(X_{m+n+1} = i_{n+1}, X_m = i_0, \dots, X_{m+n} = i_n, T = m, X_m = i)$
$\overset{\text{def. of cond. prob.}}{=} \sum_{m \geq 0} (P(\{X_{m+n+1} = i_{n+1}, X_m = i_0, \dots, X_{m+n} = i_n\} \cap \{T = m\} \mid X_m = i)$
$\cdot P(X_m = i))$

determ. by $X_0, \dots, X_m$ by def.

indep. cond. on $\{X_m = i\}$

$\overset{\text{Markov property (Th. 3.1.4)}}{=} \sum_{m \geq 0} (P(X_{m+n+1} = i_{n+1}, X_m = i_0, \dots, X_{m+n} = i_n \mid X_m = i)$
$\cdot P(T = m \mid X_m = i) \cdot P(X_m = i))$

(37) def of cond. prob., $i = i_0$

$$\sum_{m \geq 0} \Big(\overbrace{P(X_{m+n+1} = i_{n+1} \mid X_m = i_0, \dots, X_{m+n} = i_n)}^{\text{Th. 3.1.4 } p_{i_n i_{n+1}}} \cdot P(X_m = i_0, \dots, X_{m+n} = i_n \mid X_m = i) \cdot P(T = m \mid X_m = i) \cdot P(X_m = i) \Big)$$

Markov property (Th. 3.1.4)

$$p_{i_n i_{n+1}} \sum_{m \geq 0} \big(P(\{X_m = i_0, \dots, X_{m+n} = i_n\} \cap \{T = m\} \mid X_m = i) \cdot P(X_m = i) \big)$$

def. of prob.

$$p_{i_n i_{n+1}} P\Big(\bigcup_{m \geq 0} \{X_m = i_0, \dots, X_{m+n} = i_n, T = m, X_m = i\} \Big)$$

$T = m$

$$p_{i_n i_{n+1}} P\Big(\bigcup_{m \geq 0} \{X_T = i_0, \dots, X_{T+n} = i_n, T = m, X_T = i\} \Big)$$

(**)

$$p_{i_n i_{n+1}} P(X_T = i_0, \dots, X_{T+n} = i_n, T < \infty, X_T = i)$$

$\Rightarrow (*) = p_{i_n i_{n+1}}$

Similarly one shows the properties $\widehat{P}(\widehat{X}_{n+1} = i_{n+1} \mid \widehat{X}_n = i_n) = p_{i_n i_{n+1}}$ and (3.3.2).

$\Rightarrow$ proof

3.4 Recurrence and transience

Def. 3.4.1 (Recurrence / transience)
Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P.

(i) A state $i \in \mathcal{I}$ is called *recurrent*, if

$$P_i(X_n = i \text{ for infinitely many } n) = 1$$

$\longrightarrow$ the chain X_n starting in i revisits i infinitely often with prob. 1

Recall: $P_i(A) \overset{\text{def.}}{=} P(A \mid X_0 = i)$ for events A

(ii) A state $i \in \mathcal{I}$ is *transient* if

$$P_i(X_n = i \text{ for infinitely many } n) = 0$$

$\longrightarrow$ the chain X_n starting in i reenters i at most finitely often with prob. 1

Ex. 3.4.2 (Simple random walk on $\mathbb{Z}$)
Consider a Markov chain $(X_n)_{n\geq 0}$ with state space

$$\mathcal{I} = \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

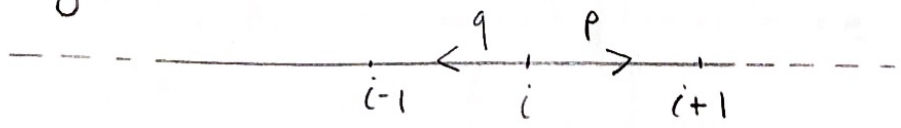
and trans. prob. given by

$$p_{i,i+1} = p \quad \text{and} \quad p_{i,i-1} = q$$

for $i \in \mathbb{Z}$,
where

$$0 < p < 1, \quad q := 1 - p$$

$\longrightarrow$ diagram:



$\longrightarrow$ represents e.g. the wealth of a gambler who on each play of the game either wins (with prob. p) or loses (with prob. q) 1 $

One shows (see later):
All states $i \in \mathcal{I}$ are recurrent, if $p = q = \frac{1}{2}$
otherwise all states are transient (i.e. if $p \neq q$)

Objective: We aim at showing that every state $i \in I$ is either recurrent or transient (see the next theorem)

For this purpose let us introduce the following random times:

Consider the first passage time to state i, i.e.

$$T_i = \inf\{n \geq 1 : X_n = i\}$$

$(\inf \emptyset \overset{\text{def}}{=} \infty)$

Define inductively the r-th passage time $T_i^{(r)}$ to i by

$$T_i^{(0)} \equiv 0 \,, \quad T_i^{(1)} = T_i \tag{3.4.1}$$

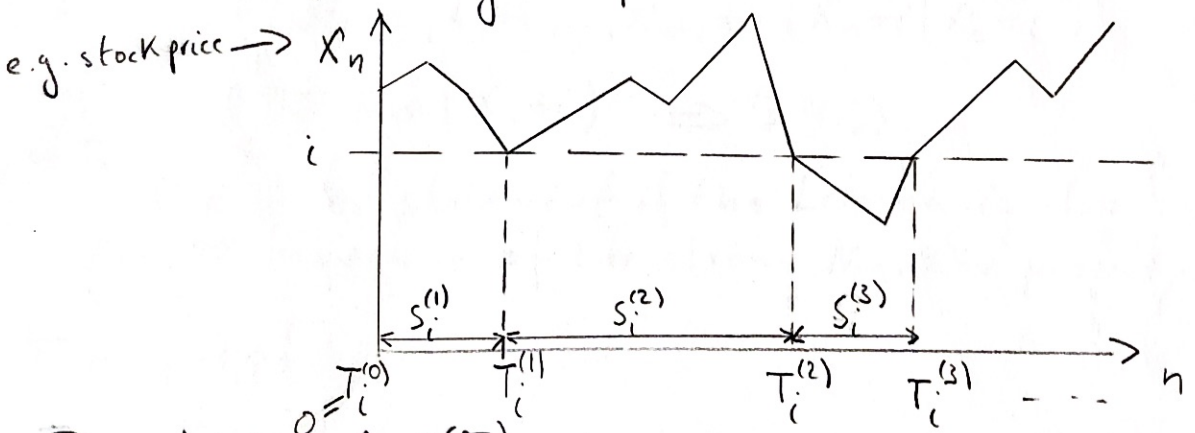
and

$$T_i^{(r+1)} = \inf\{n \geq T_i^{(r)} + 1 : X_n = i\} \,, \quad r = 0, 1, 2, \ldots$$

Further def.

$$S_i^{(r)} = \begin{cases} T_i^{(r)} - T_i^{(r-1)} \,, & \text{if } T_i^{(r-1)} < \infty \\ 0 & \text{else} \end{cases} \tag{3.4.2}$$

$\longrightarrow$ $S_i^{(r)}$ length of the r-th excursion to i



The distr. of $S_i^{(r)}$ will be important for the study of recurrence / transience.

Lemma 3.4.3 (distr. of $S_i^{(r)}$)

(conditional on $T_i^{(r-1)} < \infty$) $S_i^{(r)}$ is independent of the r.v.'s X_m, $m = 0, \ldots, T_i^{(r-1)}$ for all $r = 2, 3, \ldots$

Moreover,

$$P(S_i^{(r)} = n \mid T_i^{(r-1)} < \infty) = P_i(T_i = n) \tag{3.4.3}$$

$$\left(\overset{\text{def.}}{=} P(T_i = n \mid X_0 = i)\right)$$

Proof is based on the strong Markov property applied to the stopping time $T = T_i^{(r-1)}$

W.l.o.g. consider the case $r = 2$.

Then $S_i^{(2)}$ is the first passage time of $\tilde{X}_n := X_{T+n}$, $n \geq 0$ and $\tilde{X}_n$, $n \geq 0$ is Markov(S, i, P) under $\tilde{P}(A) := P(A \mid T < \infty, X_T = i)$ (Th. 3.3.3)

$$P(S_i^{(2)} = n \mid T_i^{(1)} < \infty) \overset{\text{def of 1. passage time}}{=} P(\tilde{X}_0 = i, \tilde{X}_1 \neq i, \dots, \tilde{X}_{n-1} \neq i, \tilde{X}_n = i \mid T < \infty, X_T = i)$$

$$\overset{\text{def of } \tilde{P}}{=} \tilde{P}(\tilde{X}_0 = i, \tilde{X}_1 \neq i, \dots, \tilde{X}_{n-1} \neq i, \tilde{X}_n = i)$$

$$= \dot{\bigcup}_{j_1, \dots, j_{n-1} \in \mathbb{N} \setminus \{i\}} \{\tilde{X}_0 = i, \tilde{X}_1 = j_1, \dots, \tilde{X}_{n-1} = j_{n-1}, \tilde{X}_n = i\}$$

$$\overset{\text{def. of prob}}{=} \sum_{j_1, \dots, j_{n-1} \in \mathbb{N} \setminus \{i\}} \underbrace{\tilde{P}(\tilde{X}_0 = i, \tilde{X}_1 = j_1, \dots, \tilde{X}_{n-1} = j_{n-1}, \tilde{X}_n = i)}_{= p_{i j_1} p_{j_1 j_2} \cdots p_{j_{n-1} i} \text{ by Th 3.1.3 and the strong Markov prop.}}$$

$$\overset{\text{Th. 3.1.3}}{=} \sum_{j_1, \dots, j_{n-1} \in \mathbb{N} \setminus \{i\}} P(X_0 = i, X_1 = j_1, \dots, X_{n-1} = j_{n-1}, X_n = i \mid X_0 = i)$$

$$\overset{\text{def of prob}}{=} P(X_0 = i, X_1 \neq i, \dots, X_{n-1} \neq i, X_n = i \mid X_0 = i)$$

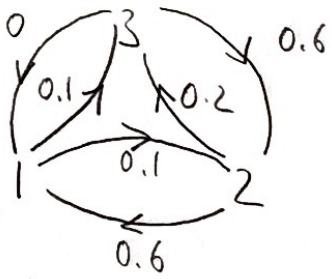
$$\overset{\text{def of } T_i}{=} P(T_i = n \mid X_0 = i) \Rightarrow (3.4.3)$$

The other statement of the Lemma is also a consequence of the strong Markov property

$\Rightarrow$ proof

Ex. 3.4.5 : Consider the car insurance of Ex. 3.2.7 with the risk categories
$1 \triangleq$ "low risk", $2 \triangleq$ "intermediate risk", $3 \triangleq$ "high risk"
X_n denotes the risk cat. of the insured in year $n \geq 0$ and is modelled by a Markov chain with $S = \{1,2,3\}$ and trans. matrix

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0.6 & 0.4 \end{pmatrix}$$



$X_0 = 1$ with prob. 1

Problem : What is the prob. that the time between two consecutive visits to the lowest risk category is smaller than 3 years ?
Or more mathematically :

$$P(S_i^{(r)} \leq 3 \text{ years} \mid T_i^{(r-1)} < \infty) \; ?$$

for $i = 1$, $r \geq 1$
By L. 3.4.3 we know that

$$P(S_i^{(r)} = n \mid T_i^{(r-1)} < \infty) = P_i(T_i = n)$$

for all i, n

$$P(S_1^{(r)} \leq 3 \mid T_1^{(r-1)} < \infty) = P_1(T_1 \leq 3)$$

($i = 1$, $n = 1,2,3$)

$P_1(T_1 \leq 3)$?

Recall : $T_i \overset{\text{def.}}{:=} \inf\{n \geq 1 : X_n = i\}$

$$\Rightarrow \quad P(T_i = n \mid X_0 = i)$$
$$= P(X_1 \neq i, X_2 \neq i, \dots, X_{n-1} \neq i, X_n = i \mid X_0 = i) \quad (*)$$

for $n \geq 1$

Note : $P_1(T_1 \leq 3) = P_1(T_1 = 1) + P_1(T_1 = 2) + P_1(T_1 = 3)$

(43) 1. $P_1(T_1=1) \overset{(*)}{=} P(X_1=1 \mid X_0=1) = P_{11} = 0.8$

2. $P_1(T_1=2) \overset{(*)}{=} P(X_1 \neq 1, X_2=1 \mid X_0=1)$

$\overset{P(X_0=1)=1}{=} P(\underbrace{X_1 \neq 1}_{X_1=2 \text{ or } 3}, X_2=1) = P(X_1=2, X_2=1)$

$+ P(X_1=3, X_2=1)$

Recall (Th. 3.1.3):

$P(X_0=i_0, X_1=i_1, \dots, X_N=i_N) \overset{\text{def}}{=} P(X_0=i_0) \, p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{N-1} i_N}$ (*)

$\Rightarrow P(X_1=2, X_2=1) \overset{P(X_0=1)=1}{=} P(X_0=1, X_1=2, X_2=1)$

and $= 1 \cdot p_{12} p_{21} = 0.1 \cdot 0.6 = 0.06$

$P(X_1=3, X_2=1) \overset{P(X_0=1)=1}{=} P(X_0=1, X_1=3, X_2=1)$

$= \overset{=\lambda_1}{1} \cdot p_{13} \cdot \overset{=0}{p_{31}} = 0$

3. $P_1(T_1=3) \overset{(*)}{=} P(\underbrace{X_1 \neq 1}_{X_1=2 \text{ or } 3}, \underbrace{X_2 \neq 1}_{X_2=2 \text{ or } 3}, X_3=1 \mid X_0=1)$

$\overset{P(X_0=1)=1}{=} P(X_0=1, X_1=2, X_2=2, X_3=1)$

$+ P(X_0=1, X_1=3, X_2=2, X_3=1)$

$+ P(X_0=1, X_1=2, X_2=3, X_3=1)$

$+ P(X_0=1, X_1=3, X_2=3, X_3=1)$

$\overset{(**)}{=} \underbrace{p_{12} \cdot p_{22} p_{21}}_{=0.012} + \underbrace{p_{13} p_{32}}_{=0.036} \cdot p_{21} + \underbrace{p_{12} \cdot p_{23} \overset{=0}{p_{31}}}_{=0} + \underbrace{p_{13} \cdot p_{33} \overset{=0}{p_{31}}}_{=0}$

$= 0.048$

$\Rightarrow P_1(T_1 \leq 3) \overset{\text{def}}{=} P(T_1 \leq 3 \mid X_0=1) = 0.8 + 0.06 + 0.048$

$= 0.908$

$\Rightarrow$ prob. that the time betw. two consecutive visits to the lowest cat. is less than 3 years is 90.8%