Proof is based on the strong Markov property applied to the stopping time $T = T_i^{(r-1)}$

W.l.o.g. consider the case $r=2$.

Then $S_i^{(2)}$ is the first passage time of $\tilde{X}_n := X_{T+n}$, $n \geq 0$ and $\tilde{X}_n$, $n \geq 0$ is Markov(δ_i, P) under $\tilde{P}(A) := P(A \mid T<\infty, X_T = i)$ (Th. 3.3.3)

$$P(S_i^{(2)} = n \mid T_i^{=T} < \infty)$$

$\overset{\text{def. of 1. passage time}}{=}$ $P(\tilde{X}_0 = i, \tilde{X}_1 \neq i, \dots, \tilde{X}_{n-1} \neq i, \tilde{X}_n = i \mid T<\infty, X_T = i)$

$\overset{\text{def of } \tilde{P}}{=}$ $\tilde{P}(\tilde{X}_0 = i, \tilde{X}_1 \neq i, \dots, \tilde{X}_{n-1} \neq i, \tilde{X}_n = i)$

$= \dot{\bigcup}_{j_1, \dots, j_{n-1} \in \mathbb{N} \setminus \{i\}} \{\tilde{X}_0 = i, \tilde{X}_1 = j_1, \dots, \tilde{X}_{n-1} = j_{n-1}, \tilde{X}_n = i\}$

$\overset{\text{def. of prob.}}{=}$ $\sum_{j_1, \dots, j_{n-1} \in \mathbb{N} \setminus \{i\}} \tilde{P}(\tilde{X}_0 = i, \tilde{X}_1 = j_1, \dots, \tilde{X}_{n-1} = j_{n-1}, \tilde{X}_n = i)$

$= p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} i}$ by Th. 3.1.3 and the strong Markov prop.

$\overset{\text{Th. 3.1.3}}{=}$ $\sum_{j_1, \dots, j_{n-1} \in \mathbb{N} \setminus \{i\}} P(X_0 = i, X_1 = j_1, \dots, X_{n-1} = j_{n-1}, X_n = i \mid X_0 = i)$

$\overset{\text{def of prob.}}{=}$ $P(X_0 = i, X_1 \neq i, \dots, X_{n-1} \neq i, X_n = i \mid X_0 = i)$

$\overset{\text{def of } T_i}{=}$ $P(T_i = n \mid X_0 = i) \Rightarrow (3.4.3)$.

The other statement of the Lemma is also a consequence of the strong Markov property

$\Rightarrow$ proof

Def. the r.v.

$$V_i = \sum_{n \geq 0} \underbrace{1_{\{X_n = i\}}}_{= \begin{cases} 1, & \text{if } X_n = i \\ 0 & \text{else} \end{cases}}$$

$\longrightarrow$ number of visits of X_n to i

Lemma 3.4.4 (distr. of V_i)

$$P_i(V_i > r) \left(\overset{\text{def.}}{=} P(V_i > r \mid X_0 = i) \right)$$

$$= (f_i)^r, \quad r = 0, 1, 2, \dots$$

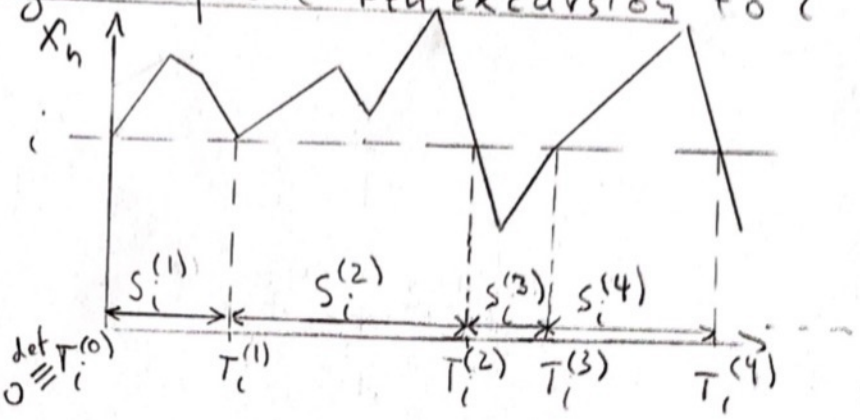
where f_i is the return probability given by

$$f_i := P_i(T_i < \infty) \quad (\overset{\text{def.}}{=} P(T_i < \infty \mid X_0 = i))$$

$\longrightarrow$ prob. that the chain starting in i ever returns to i

Proof: based on L. 3.4.3.

Recall: rth passage time $T_i^{(r)}$ to i, $S_i^{(r)}$ length of the rth excursion to i



If $X_0 = i$ then

$$\{V_i > r\} = \{T_i^{(r)} < \infty\}, \quad r \geq 0 \qquad (+)$$

(V_i: number of visits of X_n to i is bigger than r; $T_i^{(r)} < \infty$: r-th passage time to i exists)

Observe that

$$P_i(V_i > 0) \overset{\text{def}}{=} P(V_i > 0 \mid X_0 = i) = 1 = (f_i)^0$$

$\Longrightarrow$ statement of L. 3.4.4 true for $r = 0$

Induction hypothesis: statement holds for $r > 0$ (++)

$\{V_i > r+1\} \Longrightarrow \{T_i^{(r+1)} < \infty\}$ if $X_0 = i$

$$P(V_i > r+1 \mid X_0 = i) = P(T_i^{(r+1)} < \infty \mid X_0 = i)$$

$T_i^{(r+1)} = T_i^{(r)} + S_i^{(r+1)}$

$$= P(T_i^{(r)} < \infty, S_i^{(r+1)} < \infty \mid X_0 = i)$$

$$\overset{\text{def}}{=} P_i(T_i^{(r)} < \infty, S_i^{(r+1)} < \infty)$$

$$= \frac{P_i(T_i^{(r)} < \infty, S_i^{(r+1)} < \infty)}{P_i(T_i^{(r)} < \infty)} \cdot P_i(T_i^{(r)} < \infty)$$

$\overset{\text{def. of cond. expect.}}{=} P_i(S_i^{(r+1)} < \infty \mid T_i^{(r)} < \infty) \cdot P_i(T_i^{(r)} < \infty) \qquad (+++)$

Note that

$$P_i(S_i^{(r+1)} < \infty \mid T_i^{(r)} < \infty) = P_i\Big(\dot{\bigcup_{n \geq 0}} \{S_i^{(r+1)} = n\} \mid T_i^{(r)} < \infty\Big)$$

($\dot{\cup}$: disjoint)

$$= \sum_{n \geq 0} P_i(S_i^{(r+1)} = n \mid T_i^{(r)} < \infty) \overset{\text{L. 3.4.3}}{=} \sum_{n \geq 0} P_i(T_i = n)$$

$= P_i(T_i < \infty) \overset{\text{def.}}{=} f_i$

On the other hand

$P_i(T_i^{(r)} < \infty) \overset{\text{def}}{=} P(T_i^{(r)} < \infty \mid X_0 = i) \overset{(\dagger)}{=}$

$P(V_i > r \mid X_0 = i) \overset{\text{induction hypothesis } (\dagger\dagger)}{=} (f_i)^r$

$\overset{(\dagger\dagger\dagger)}{\Rightarrow} P(V_i > r+1 \mid X_0 = i) = f_i \cdot (f_i)^r = (f_i)^{r+1}$

$\Rightarrow$ induction $\Rightarrow$ statement of L. 3.4.4 valid for all $r \geq 0$

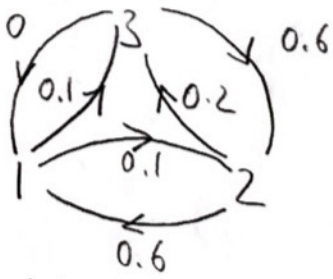
$\Rightarrow$ proof

Ex. 3.4.5 : Consider the car insurance of Ex. 3.2.7 with the risk categories

$1 \triangleq$ "low risk", $2 \triangleq$ "intermediate risk", $3 \triangleq$ "high risk"

X_n denotes the risk cat. of the insured in year $n \geq 0$ and is modelled by a Markov chain with $S = \{1,2,3\}$ and trans. matrix

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0.6 & 0.4 \end{pmatrix}$$



$X_0 = 1$ with prob. 1

Problem : What is the prob. that the time between two consecutive visits to the lowest risk category is smaller than 3 years ?

Or more mathematically :

$P(S_i^{(r)} \leq 3 \text{ years} \mid T_i^{(r-1)} < \infty)$?

for $i = 1$, $r \geq 1$

By L. 3.4.3 we know that

$$P(S_i^{(r)} = n \mid T_i^{(r-1)} < \infty) = P_i(T_i = n)$$

for all i, n

$\overset{i=1}{\Rightarrow}$ $P(S_1^{(r)} \leq 3 \mid T_1^{(r-1)} < \infty) = P_1(T_1 \leq 3)$

$n = 1,2,3$

$P_1(T_1 \leq 3)$?

Recall : $T_i \overset{\text{def.}}{=} \inf\{n \geq 1 : X_n = i\}$

$\Rightarrow P(T_i = n \mid X_0 = i)$

$= P(X_1 \neq i, X_2 \neq i, \dots, X_{n-1} \neq i, X_n = i \mid X_0 = i)$ (*)

for $n \geq 1$

Note : $P_1(T_1 \leq 3) = P_1(T_1 = 1) + P_1(T_1 = 2) + P_1(T_1 = 3)$

(43) 1. $P_1(T_1=1) \overset{(*)}{=} P(X_1=1 \mid X_0=1) = P_{11} = 0.8$

2. $P_1(T_1=2) \overset{(*)}{=} P(X_1 \neq 1, X_2=1 \mid X_0=1)$

$P(X_0=1)=1$ $= P(\underbrace{X_1 \neq 1}_{X_1=2 \text{ or } 3}, X_2=1) = P(X_1=2, X_2=1)$

$+ P(X_1=3, X_2=1)$

Recall Th. 3.1.3:

$P(X_0=i_0, X_1=i_1, \dots, X_N=i_N) \overset{\text{def}}{=} P(X_0=i_0)\, p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{N-1} i_N}$ (**)

$\Rightarrow P(X_1=2, X_2=1) \overset{P(X_0=1)=1}{=} P(X_0=1, X_1=2, X_2=1)$

and $= 1 \cdot p_{12} p_{21} = 0.1 \cdot 0.6 = 0.06$

$P(X_1=3, X_2=1) \overset{P(X_0=1)=1}{=} P(X_0=1, X_1=3, X_2=1)$

$= \overset{=\lambda_1}{1} p_{13} \cdot \overset{=0}{p_{31}} = 0$

3. $P_1(T_1=3) \overset{(*)}{=} P(\underbrace{X_1 \neq 1}_{X_1=2 \text{ or } 3}, \underbrace{X_2 \neq 1}_{X_2=2 \text{ or } 3}, X_3=1 \mid X_0=1)$

$P(X_0=1)=1$ $= P(X_0=1, X_1=2, X_2=2, X_3=1)$

$+ P(X_0=1, X_1=3, X_2=2, X_3=1)$

$+ P(X_0=1, X_1=2, X_2=3, X_3=1)$

$+ P(X_0=1, X_1=3, X_2=3, X_3=1)$

$\overset{(**)}{=} \underbrace{p_{12} \cdot p_{22} p_{21}}_{=0.012} + \underbrace{p_{13} p_{32}}_{=0.036} \cdot p_{21} + \underbrace{p_{12} \cdot p_{23} \overset{=0}{p_{31}}}_{=0} + \underbrace{p_{13} \cdot p_{33} \overset{=0}{p_{31}}}_{=0}$

$= 0.048$

$\Rightarrow P_1(T_1 \leq 3) \overset{\text{def}}{=} P(T_1 \leq 3 \mid X_0=1) = 0.8 + 0.06 + 0.048$

$= 0.908$

$\Rightarrow$ prob. that the time betw. two consecutive visits to the lowest cat. is less than 3 years is 90.8%

Th. 3.4.6 (Dichotomy of states)

Recall from Th. 3.1.5 that $P(X_n=j \mid X_0=i) = p_{ij}^{(n)}$.

Then either the statement

(i) If $P_i(T_i < \infty) = 1$, then i is recurrent and

$$\sum_{n \geq 0} p_{ii}^{(n)} = \infty$$

or

(ii) If $P_i(T_i < \infty) < 1$, then i is transient and

$$\sum_{n \geq 0} p_{ii}^{(n)} < \infty$$

hold.

In particular, every state of $(X_n)_{n \geq 0}$ either transient or recurrent

(44) **Proof:**

(i) $P(T_i < \infty \mid X_0 = i) = 1$

$\overset{\text{def. } f_i}{\Longleftrightarrow}$

We know by L. 3.4.4 that

$$P(V_i > r \mid X_0 = i) = (f_i)^r$$

$\overset{f_i = 1}{\Longrightarrow}$

$$P(V_i > r \mid X_0 = i) = (1)^r = 1$$

for all $r \geq 0$

$\overset{r \to \infty}{\Longrightarrow}$

$$P(V_i = \infty \mid X_0 = i) = \lim_{r \to \infty} P(V_i > r \mid X_0 = i)$$

$$= \lim_{r \to \infty} 1 = 1$$

$\Longrightarrow$ number of visits to i is ∞

$\overset{\text{Def. 3.4.1}}{\Longrightarrow}$ i recurrent

Since $P(V_i = \infty \mid X_0 = i) = 1$ it follows that

$$E_i[V_i] \overset{\text{def}}{=} E[V_i \mid X_0 = i] = \infty \quad (+)$$

Note that

$$V_i \overset{\text{def}}{=} \sum_{n \geq 0} \underbrace{1_{\{X_n = i\}}}_{= \begin{cases} 1, & X_n = i \\ 0 & \text{else} \end{cases}}$$

$$\Rightarrow E[V_i \mid X_0 = i] = E\left[\sum_{n \geq 0} 1_{\{X_n = i\}} \mid X_0 = i\right]$$

$$= \sum_{n \geq 0} \underbrace{E[1_{\{X_n = i\}} \mid X_0 = i]}_{= P(\{X_n = i\} \mid X_0 = i)} = \sum_{n \geq 0} P(X_n = i \mid X_0 = i) \quad (++)$$

Recall:

$E[1_A \mid B] \overset{!}{=} P(A \mid B)$ for events A, B

$1_A \overset{\text{def}}{=} \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0 & \text{else} \end{cases}$

It follows from Th. 3.1.5 (ii) that

$$P(X_n = i \mid X_0 = i) = p_{ii}^{(n)}$$

where $p_{ij}^{(n)} \mid i, j \in I$ are entries of the matrix

$$A := P^n = \left(p_{ij}^{(n)}\right)_{i,j \in I}$$

$\overset{(+), (++)}{\Longrightarrow}$

$$\infty = E_i[V_i] = \sum_{n \geq 0} p_{ii}^{(n)} \Longrightarrow (i)$$

(95) (ii) $f_i \overset{\text{def}}{=} P(T_i < \infty \mid X_0 = i) < 1$

It follows that

$$\sum_{r=0}^{\infty} P(V_i > r \mid X_0 = i) = \sum_{r=0}^{\infty} \sum_{v=r+1}^{\infty} P(V_i = v \mid X_0 = i)$$

$$= \sum_{r=0}^{\infty} \sum_{v=r+1}^{\infty} E[\, 1_{\{V_i = v\}} \mid X_0 = i] =$$

$$E\Big[\sum_{r=0}^{\infty} \sum_{v=r+1}^{\infty} 1_{\{V_i = v\}} \mid X_0 = i \Big]$$

$$= E\Big[\sum_{r=0}^{\infty} \sum_{v=1}^{\infty} 1_{\{V_i = v,\, v \geq r+1\}} \mid X_0 = i \Big]$$

$$= E\Big[\sum_{v=1}^{\infty} \sum_{r=0}^{v-1} 1_{\{V_i = v,\, v \geq r+1\}} \mid X_0 = i \Big]$$

$$= E\Big[\sum_{v=1}^{\infty} \sum_{r=0}^{v-1} 1_{\{V_i = v\}} \mid X_0 = i \Big] = \sum_{v=1}^{\infty} \sum_{r=0}^{v-1} \underbrace{E[\, 1_{\{V_i = v\}} \mid X_0 = i]}_{P(V_i = v \mid X_0 = i)}$$

$$= \sum_{v=1}^{\infty} \underbrace{\sum_{r=0}^{v-1} P(V_i = v \mid X_0 = i)}_{= v \cdot P(V_i = v \mid X_0 = i)} = \sum_{v=1}^{\infty} v \cdot P(V_i = v \mid X_0 = i)$$

$$\overset{\text{def. of expectations}}{=} E[V_i \mid X_0 = i] \overset{(i)}{=} \sum_{n \geq 0} p_{ii}^{(n)} \qquad (*)$$

On the other hand we know by L. 3.4.3 that

$$P(V_i > r \mid X_0 = i) = f_i^r \text{ for all } r \geq 0$$

$$\Rightarrow \sum_{r=0}^{\infty} P(V_i > r \mid X_0 = i) = \sum_{r=0}^{\infty} (f_i)^r \overset{0 \leq f_i < 1}{=} \frac{1}{1 - f_i} \qquad (**)$$

geom. sum

$< \infty$! Recall: geometrical sum

$$\sum_{n \geq 0} q^n = \frac{1}{1-q}, \text{ if } 0 \leq q < 1$$

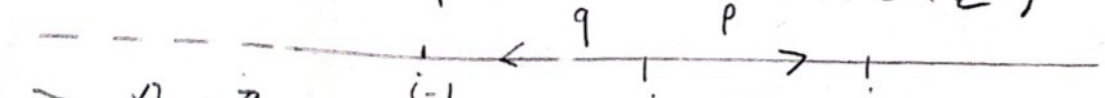
$$\overset{(*),(**)}{\Rightarrow} E[V_i \mid X_0 = i] = \sum_{n \geq 0} p_{ii}^{(n)} = \frac{1}{1 - f_i} < \infty$$

$$\Rightarrow P(V_i = \infty \mid X_0 = i) = 0$$

$\Rightarrow$ number of visits to i is at most finitely often with prob. 1 $\Rightarrow$ i transient $\Rightarrow$ (ii) $\Rightarrow$ proof

Let us revisit

Ex. 3.4.2 (simple random walk on $\mathbb{Z}$)



$(0 < p < 1,\ q := 1 - p)$

$\rightarrow$ $X_n \in \mathbb{Z}$ e.g. the wealth of a gambler on the n-th play of a game (where he/she can either win (with prob. p) or lose (with prob. q)) ($X_n < 0$ means loan)

We know that $0 \in \mathcal{I} = \mathbb{Z}$ is recurrent (transient) $\overset{\text{Th. 3.4.6}}{\Longleftrightarrow}$

$$\sum_{n \geq 0} p_{00}^{(n)} = \infty \quad \left(\sum_{n \geq 0} p_{00}^{(n)} < \infty \right)$$

where $P(X_n = 0 \mid X_0 = 0) \overset{\text{Th. 3.1.5}}{=} p_{00}^{(n)}$

It is obvious that X_n cannot return to 0 after odd steps, i.e. $n = 2m+1$ for $m \geq 0$

$$\Rightarrow \quad P(X_{2m+1} = 0 \mid X_0 = 0) = p_{00}^{(2m+1)} = 0 \qquad (*)$$

for all $m \geq 0$

Suppose that n is even, i.e. $n = 2m$ for $m \geq 0$

Then X_n can only hit 0, if there were m steps up and m steps down on the real line.

The prob. for such a random walk is $p^m \cdot q^m$

Further, there ex. $\binom{2m}{m}$ such random walks.

$$\Rightarrow p_{00}^{(2m)} = P(X_{2m} = 0 \mid X_0 = 0) = \binom{2m}{m} p^m q^m \qquad (+)$$