We know that $0 \in \mathbb{J} = \mathbb{Z}$ is recurrent (transient) $\overset{\text{Th. 3.4.6}}{\Longleftrightarrow}$

$$\sum_{n \geq 0} p_{00}^{(n)} = \infty \quad \left(\sum_{n \geq 0} p_{00}^{(n)} < \infty \right)$$

where $P(X_n = 0 \mid X_0 = 0) \overset{\text{Th. 3.1.5}}{=} p_{00}^{(n)}$

It is obvious that X_n cannot return to 0 after odd steps, i.e. $n = 2m+1$ for $m \geq 0$

$$\Rightarrow \quad P(X_{2m+1} = 0 \mid X_0 = 0) = p_{00}^{(2m+1)} = 0 \qquad (*)$$

for all $m \geq 0$

Suppose that n is even, i.e. $n = 2m$ for $m \geq 0$

Then X_n can only hit 0, if there were m steps up and m steps down on the real line.

The prob. for such a random walk is $p^m \cdot q^m$

Further, there ex. $\binom{2m}{m}$ such random walks.

$$\Rightarrow p_{00}^{(2m)} = P(X_{2m} = 0 \mid X_0 = 0) = \binom{2m}{m} p^m q^m \qquad (+)$$

Recall: $\binom{n}{k} \overset{\text{def.}}{=} \frac{n!}{(n-k)!\, k!}$ $\quad n! \overset{\text{def}}{=} n \cdot (n-1)(n-2) \cdots 2 \cdot 1$

Use the following fact (Stirling's formula):

$$n! \sim \sqrt{2\pi n}\, (n/e)^n \quad \text{as } n \to \infty \qquad (++)$$

where $a_n \sim b_n$ means $\frac{a_n}{b_n} \longrightarrow 1$

See intr. to basic analysis.

$$\overset{(+),(++)}{\Longrightarrow} \quad p_{00}^{(2m)} = \binom{2m}{m} p^m \cdot q^m = \frac{(2m)!}{m!\, m!} (p \cdot q)^m$$

$$\sim \frac{\sqrt{2\pi \cdot 2m}\, (2m/e)^{2m}}{\left(\sqrt{2\pi m}\, (m/e)^m\right)^2} (p \cdot q)^m = \frac{1}{\sqrt{\pi m}} (4 \cdot p \cdot q)^m \qquad (+++)$$

for $m \to \infty$

1. case: $p = q \overset{(+++)}{\Longrightarrow} p = q = \frac{1}{2}$

$\Rightarrow$ ex. N s.t. for all $m \geq N$

$$p_{00}^{(2m)} \geq \frac{1}{2} \frac{1}{\sqrt{\pi m}} \left(4 \cdot \frac{1}{2} \cdot \frac{1}{2}\right)^m = \frac{1}{2} \frac{1}{\sqrt{\pi m}} = \frac{1}{2} \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{m}}$$

$$\Rightarrow \sum_{n \geq 0} p_{00}^{(n)} \overset{(*)}{=} \sum_{m \geq 0} p_{00}^{(2m)} \geq \sum_{m \geq N} p_{00}^{(2m)} \geq \sum_{m \geq N} \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{m}} = \infty$$

$$\Rightarrow \sum_{n \geq 0} p_{00}^{(n)} = \infty \Rightarrow \text{state 0 recurrent} \Rightarrow \text{the gambler starting with wealth zero will have wealth zero infinitely often with prob. 1}$$

2. case: $p \neq q \Rightarrow r := 4pq < 1$

Because of (+++) there ex. N s.t. for all $m \geq N$

$$p_{00}^{(2m)} \leq \frac{3}{2} \frac{1}{\sqrt{\pi m}} \cdot \underbrace{(4 \cdot p \cdot q)^m}_{= r} \leq \frac{3}{2} \frac{1}{\sqrt{\pi}} \cdot r^m$$

(47) $\Rightarrow \sum_{m \geq N} p_{00}^{(2m)} \leq \frac{3}{2} \frac{1}{\sqrt{\pi}} \sum_{m \geq N} r^m \leq \frac{3}{2} \frac{1}{\sqrt{\pi}} \sum_{m \geq 0} r^m \overset{\text{geom. sum}}{=} \frac{3}{2} \frac{1}{\sqrt{\pi}} \frac{1}{1-r} < \infty$

$\Rightarrow \sum_{n \geq 0} p_{00}^{(n)} \overset{(*)}{=} \sum_{m \geq 0} p_{00}^{(2m)} < \infty$

$\Rightarrow$ state 0 transient $\Rightarrow$ the gambler with initial wealth zero will have wealth zero at most finitely often with prob. 1

Recall:

(i) i leads to j (or $i \to j$) $\iff$
$P(X_n = j \text{ for some } n \geq 0 \mid X_0 = i) > 0$

(ii) i communicates with j (or $i \leftrightarrow j$)
if $i \to j$ and $j \to i$

(iii) $C \subseteq I$ communicating class $\iff$
There ex. $j \in I$:
$C = \{i \in I : i \leftrightarrow j\}$

(iv) C in (iii) is called closed, if
$i \in C$ and $i \to j \implies j \in C$

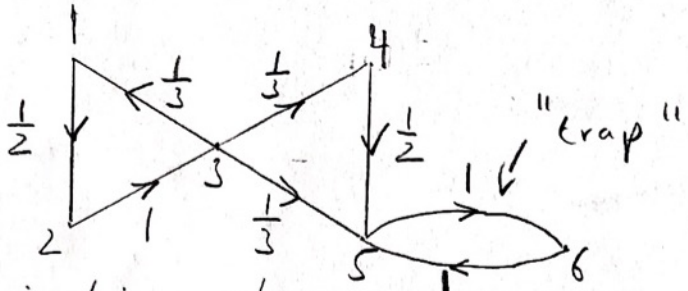
Rem. 3.4.3 (Criterion for $i \to j$)

$i \to j \iff$ there ex. $n \geq 0$ s.t. $p_{ij}^{(n)} > 0$

This follows from Th. 3.1.5 (ii)

$\Rightarrow$ $i \leftrightarrow j \iff$ there ex. $n, m \geq 0$ s.t.
$p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$

Ex.:

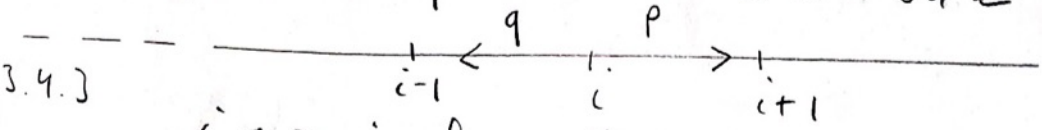


Rem. 3.4.3 $\longrightarrow$ communicating classes: $\{1,2,3\}, \{4\}, \{5,6\}$ (closed)

Th. 3.4.4: If C is a communicating class, then either all $i \in C$ are transient or recurrent

Proof: Consequence of Rem. 3.4.3

Ex. 3.4.5: simple random walk on $\mathbb{Z}$



$(0 < p < 1, q = 1-p)$

Rem. 3.4.3 $\Rightarrow$ $i \leftrightarrow j$ for all $i, j \in \mathbb{Z}$

$\Rightarrow$ $C = I$ is the only communicating class

We know from Ex. 3.4.2 that

Th. 3.4.4 $\Rightarrow$ all $i \in I \overset{\text{def}}{=} \mathbb{Z}$ are recurrent, if $p = q = \frac{1}{2}$
and transient otherwise

Rem. 3.4.6:

(i) Every recurrent class is closed

(ii) Every finite closed class is recurrent

See Norris

Def. 3.4.7 (irreducible chain)

A Markov chain is called *irreducible*, if the state space S is a communicating class

Ex: random walk in $\mathbb{Z}$

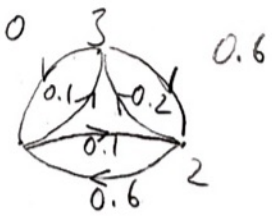
Th. 3.4.8: Let $X_n, n \geq 0$ be irreducible and recurrent. Then for all $j \in S$:

$$P(T_j < \infty) = 1$$

($T_j \overset{\text{def.}}{=} \inf\{n \geq 1 : X_n = j\}$)

Proof: Consequence of Th. 3.4.6 (dichotomy of states).

Ex: car insurance of Ex. 3.2.7



$\Rightarrow$ $X_n, n \geq 0$ irreducible

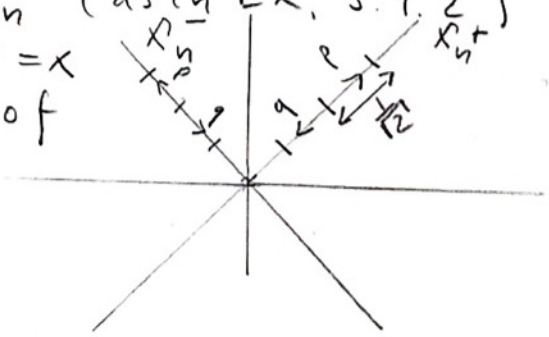
$\Rightarrow$ $S = \{1, 2, 3\}$ closed $\overset{\text{Rem. 3.4.6(ii)}}{\Rightarrow}$ S recurrent

$\overset{\text{Th. 3.4.8}}{\Rightarrow}$ $P(T_j < \infty) = 1$

$\rightarrow$ Prob. that a risk category is ever visited by the insured is one

Ex. 3.4.9 (Simple symmetric rand. walk on $\mathbb{Z}^2$)

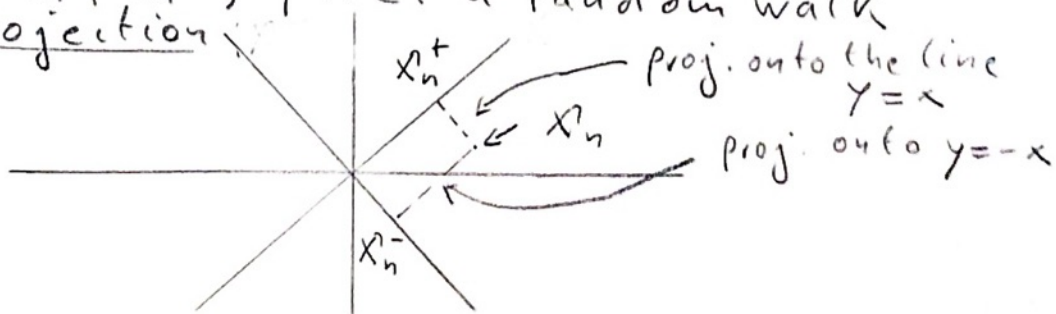
Consider a rand. walk X_n^+ (as in Ex. 3.4.2) on the diagonal line $y = x$ with $p = q = \frac{1}{2}$ and steps of length $\frac{1}{\sqrt{2}}$



Similarly, consider X_n^- on $y = -x$

Assumption: X_n^+, X_n^- independent

We use X_n^+ and X_n^- to construct a new Markov chain $X_n = (X_n^{(1)}, X_n^{(2)})$, i.e. a random walk on $\mathbb{Z}^2$ by *projection*:

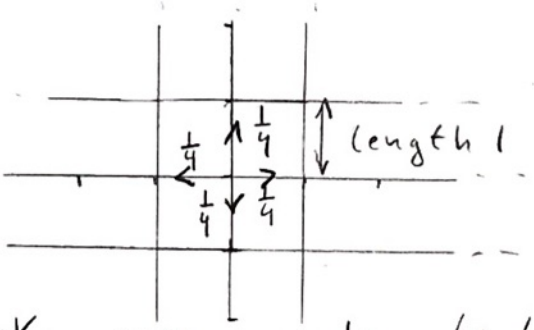


Pythagoras $\Rightarrow$ $\underbrace{\|X_n\|^2}_{\text{def}} = |X_n^+|^2 + |X_n^-|^2$ (*)

$\stackrel{\text{def}}{=} |X_n^{(1)}|^2 + |X_n^{(2)}|^2$

X_n^+, X_n^- indep. $\Rightarrow$ X_n, $n \geq 0$ Markov chain with state space $\mathcal{I} = \mathbb{Z}^2$

def $\{(x_1, x_2) : x_1, x_2 \in \mathbb{Z}\}$ of the form



$\rightarrow$ a drunk man wanders to the south, west, east or north with prob. $\frac{1}{4}$

$\rightarrow$ trans. prob. $p_{ij} = \begin{cases} \frac{1}{4} & \text{if } \|i-j\| = 1 \\ 0 & \text{else} \end{cases}$ ($\|i-j\|$: distance between i and j in $\mathbb{Z}^2$)

Because of (*) we see that

$X_n = 0 \iff X_n^+ = 0$ and $X_n^- = 0$ (**)

Ex. 3.4.2 shows that

$P_0(X_{2m+1}^{\pm} = 0) = 0$, $m \geq 0$

and $P_0(X_{2m}^{\pm} = 0) = \binom{2m}{m}\left(\frac{1}{2}\right)^m \cdot \left(\frac{1}{2}\right)^m = \binom{2m}{m}\left(\frac{1}{2}\right)^{2m}$

$\Rightarrow$ $p_{00}^{(2m+1)} \stackrel{\text{Th. 3.1.5}}{=} P_0(X_{2m+1} = 0) \stackrel{(**)}{=} P_0(X_{2m+1}^+ = 0 \text{ and } X_{2m+1}^- = 0) = 0$

and $p_{00}^{(2m)} \stackrel{\text{Th. 3.1.5}}{=} P_0(X_{2m} = 0) \stackrel{(**)}{=} P_0(X_{2m}^+ = 0 \text{ and } X_{2m}^- = 0)$

X_n^+, X_n^- indep. $P_0(X_{2m}^+ = 0) \cdot P_0(X_{2m}^- = 0) = \left(\binom{2m}{m}\left(\frac{1}{2}\right)^{2m}\right)^2$

Ex. 3.4.2 (+++) $\sim \frac{1}{\pi} \cdot \frac{1}{m}$ for $m \to \infty$

$\Rightarrow \sum_{n \geq 0} p_{00}^{(n)} = \sum_{m \geq 1} \underbrace{p_{00}^{(2m)}}_{\sim \frac{1}{\pi}\cdot\frac{1}{m}} = \infty$

Th. 3.4.6 $\Rightarrow$ 0 recurrent

$\mathcal{I} = \mathbb{Z}^2$ communicating class (i.e. X_n irreducible)

Th. 3.4.4 $\Rightarrow$ all states in $\mathbb{Z}^2$ are recurrent

3.5 Invariant distributions

—> useful to study e.g. the long-time behaviour of distributions of stock prices $S_n, n \geq 0$

Def. 3.5.1 (Invariant distribution)
Let $P = (p_{ij})_{i,j \in I}$ be the trans. matrix of a Markov chain. Further, let $\lambda = (\lambda_i)_{i \in I}$ be a sequence with $\lambda_i \geq 0$ for all $i \in I$. Then λ is called invariant for P if

$$\lambda P = \lambda \qquad (3.5.1)$$

$$\left(\Leftrightarrow \quad (\lambda P)_j \overset{\text{def}}{=} \sum_{i \in I} \lambda_i p_{ij} = \lambda_j \text{ for all } j \in I \right)$$

If in addition

$$\sum_{i \in I} \lambda_i = 1$$

holds. Then we say that λ is an invariant distribution

Ex. 3.5.2: $I = \{1,2\}$, $P = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $\lambda = (\lambda_1, \lambda_2)$

$(\lambda P) = ((\lambda P)_1, (\lambda P)_2)$

$(\lambda P)_1 = \lambda_1 \cdot \frac{2}{3} + \lambda_2 \cdot \frac{1}{2}$, $(\lambda P)_2 = \lambda_1 \frac{1}{3} + \lambda_2 \cdot \frac{1}{2}$

$$\Rightarrow \quad \lambda P = \lambda \iff \begin{cases} \lambda_1 \frac{2}{3} + \lambda_2 \frac{1}{2} = \lambda_1 & (+) \\ \lambda_1 \frac{1}{3} + \lambda_2 \cdot \frac{1}{2} = \lambda_2 \end{cases}$$

$\lambda = (\lambda_1, \lambda_2)$ invariant distr. if (+) and

$$\lambda_1 + \lambda_2 = 1 \qquad (++)$$

holds

From (+) it follows that

$$\lambda_2 = \frac{2}{3} \lambda_1$$

$$\overset{(++)}{\Rightarrow} \quad \lambda_1 + \frac{2}{3}\lambda_1 = 1 \Rightarrow \lambda_1 = \frac{1}{1 + \frac{2}{3}} = \frac{1}{2} \cdot \frac{1}{\frac{1}{2} + \frac{1}{3}} = 0.6$$

and

$$\lambda_2 = \frac{2}{3} \cdot \frac{1}{2} \left(\frac{1}{\frac{1}{2} + \frac{1}{3}}\right) = \frac{1}{3} \cdot \frac{1}{\left(\frac{1}{2} + \frac{1}{3}\right)} = 0.4$$

Invariance of a sequence $\lambda = (\lambda_i)_{i \in I}$ is linked to the terms equilibrium and stationary

"stationary" is meant in the sense of init. distr., i.e. $P(X_0 = i) = \lambda_i$

Th. 3.5.3: Let $X_n, n \geq 0$ be Markov(λ, P). Assume that λ is invariant for P. Then $Y_n := X_{m+n}, n \geq 0$ is also Markov(λ, P)

Rem: Even after a time-shift the new Markov chain (i.e. Y_n) has the same initial distr. $\lambda \longrightarrow$ "stationary"

Proof: (consequence of the Markov property

" equilibrium " is meant in the sense of

Th. 3.5.4: Let $\mathcal{I}$ be a finite state space.

Assume for some $i \in \mathcal{I}$ that

$$p_{ij}^{(n)} \xrightarrow[n\to\infty]{} \pi_j \text{ for all } j \in \mathcal{I}$$

Recall: $P^n = (p_{ij}^{(n)})_{i,j\in\mathcal{I}}$

Then $\pi = (\pi_j)_{j\in\mathcal{I}}$ is an invariant distr.

Rem.: Remember that $P(X_n = j \mid X_0 = i) \overset{\text{Th. 3.1.5}}{=} p_{ij}^{(n)} \longrightarrow \pi_j$.

$\Rightarrow$ for large n the distr. of X_n is in an "equilibrium" ↖ likewise

Proof: Since $\mathcal{I}$ is finite we see that

$$\underset{\text{sum}}{\text{finite}} \rightarrow \sum_{j\in\mathcal{I}} \pi_j = \sum_{j\in\mathcal{I}} \lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} \underbrace{\left(\sum_{j\in\mathcal{I}} p_{ij}^{(n)}\right)}_{=1} = 1$$

Note: $P^{n+1} = (P^n)\cdot P = (p_{ij}^{(n)})_{i,j\in\mathcal{I}} \cdot (p_{ij})_{i,j\in\mathcal{I}}$

$\overset{\text{def}}{=} (p_{ij}^{(n+1)})_{i,j\in\mathcal{I}}$

$\overset{\text{matrix multipl.}}{\Rightarrow}$ $p_{ij}^{(n+1)} = \sum_{k\in\mathcal{I}} p_{ik}^{(n)} \cdot p_{kj}$ finite sum

$$\Rightarrow \quad \pi_j = \lim_{n\to\infty} p_{ij}^{(n+1)} = \lim_{n\to\infty} \sum_{k\in\mathcal{I}} p_{ik}^{(n)} \cdot p_{kj}$$

$$= \sum_{k\in\mathcal{I}} \lim_{n\to\infty} p_{ik}^{(n)} \cdot p_{kj} = \sum_{k\in\mathcal{I}} \pi_k \cdot p_{kj} \quad \text{for all } j \in \mathcal{I}$$

$\Leftrightarrow$ $\pi P = \pi$ $\Rightarrow$ proof.