Rem: Even after a time-shift the new Markov chain (i.e. Y_n) has the same initial distr. $\lambda \longrightarrow$ "stationary"

Proof: (consequence of the Markov property

"equilibrium" is meant in the sense of

Th. 3.5.4: Let $\mathcal{I}$ be a finite state space.
Assume for some $i \in \mathcal{I}$ that

$$p_{ij}^{(n)} \xrightarrow[n\to\infty]{} \pi_j \text{ for all } j \in \mathcal{I}$$

Recall: $P^n = (p_{ij}^{(n)})_{i,j\in\mathcal{I}}$

Then $\pi = (\pi_j)_{j\in\mathcal{I}}$ is an invariant distr.

Rem.: Remember that $P(X_n = j \mid X_0 = i) \overset{\text{Th. 3.1.5}}{=} p_{ij}^{(n)} \longrightarrow \pi_j$.
$\Rightarrow$ for large n the distr. of X_n is in an "equilibrium" (likewise)

Proof: Since $\mathcal{I}$ is finite we see that

$$\underset{\text{finite sum}}{\to} \sum_{j\in\mathcal{I}} \pi_j = \sum_{j\in\mathcal{I}} \lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} \underbrace{\left(\sum_{j\in\mathcal{I}} p_{ij}^{(n)}\right)}_{=1} = 1$$

Note: $P^{n+1} \overset{\text{def}}{=} (p_{ij}^{(n+1)})_{i,j\in\mathcal{I}} = (P^n)\cdot P = (p_{ij}^{(n)})_{i,j\in\mathcal{I}} \cdot (p_{ij})_{i,j\in\mathcal{I}}$

matrix multipl. $\Rightarrow$ $p_{ij}^{(n+1)} = \sum_{k\in\mathcal{I}} p_{ik}^{(n)} \cdot p_{kj}$ (finite sum)

$$\Rightarrow \pi_j = \lim_{n\to\infty} p_{ij}^{(n+1)} = \lim_{n\to\infty} \sum_{k\in\mathcal{I}} p_{ik}^{(n)} \cdot p_{kj}$$

$$= \sum_{k\in\mathcal{I}} \lim_{n\to\infty} p_{ik}^{(n)} \cdot p_{kj} = \sum_{k\in\mathcal{I}} \pi_k \cdot p_{kj} \text{ for all } j \in \mathcal{I}$$

$\Leftrightarrow$ $\pi P = \pi$ $\Rightarrow$ proof.

Def. 3.5.5 (positive/null recurrent states)
Define the expected return time m_i by

$$m_i = E[T_i \mid X_0 = i], \quad i \in \mathcal{I}$$

A state $i \in \mathcal{I}$ is called positive recurrent, if $m_i < \infty$

A recurrent state i with $m_i = \infty$ is called null recurrent

Rem. 3.5.6: If $m_i < \infty \Rightarrow P(T_i < \infty \mid X_0 = i) = 1$
$\overset{\text{Th. 3.4.6}}{\Leftarrow}$ i recurrent $\Rightarrow$ i positive recurrent $\Rightarrow$ i recurrent
However: i recurrent $\not\Rightarrow$ i positive recurrent

Th. 3.5.7: Let $X_n, n\geq 0$ be an irreducible chain with trans. matrix P. Then the following statements are equivalent
(i) every state is positive recurrent
(ii) some state i is positive recurrent

(iii) P has an invariant distribution π

Moreover, when (iii) is valid we have that

$$m_i = \frac{1}{\pi_i} \quad \text{for all } i \in S \qquad (3.5.2)$$

Ex. 3.5.8: Consider the car insurance of Ex. 3.2.7

X_n risk category of the customer in year n

$\longrightarrow$ $X_n, n \geq 0$ modelled by a Markov chain with $S = \{1,2,3\}$, initial distr. $P(X_0 = 1) = 1$ and

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0.6 & 0.4 \end{pmatrix} \qquad (*)$$

Ex. after Th 3.4.8 $\Longrightarrow$ $X_n, n \geq 0$ irreducible

Problem: What is the mean time between two consecutive visits of the customer to the lowest risk category?

Or more precisely: $(r-1)$-th passage time

$$E[S_i^{(r)} \mid T_i^{(r-1)} < \infty] \; ?$$

(length of the r-th excursion to i)

for $i = 1$, $r \geq 1$

From Lemma 3.4.3 we know that

$$P(S_i^{(r)} = n \mid T_i^{(r-1)} < \infty) = P(T_i = n \mid X_0 = i), \; n \geq 0$$

Def of expect. $\Longrightarrow$

$$E[S_i^{(r)} \mid T_i^{(r-1)} < \infty] = E[T_i \mid X_0 = i]$$

$i = 1$, $P(X_0 = 1) = 1$ $\Longrightarrow$

$$E[S_1^{(r)} \mid T_1^{(r-1)} < \infty] = E[T_1 \mid X_0 = 1] = E[T_1]$$

$E[T_1]$? $\rightarrow$ use the last theorem, i.e. find a $\pi = (\pi_1, \pi_2, \pi_3)$ s.t.

$(*)$ $\Longleftrightarrow$

$$\pi P = \pi \text{ and } \pi_1 + \pi_2 + \pi_3 = 1$$

$$\begin{cases} 0.8\pi_1 + 0.6\pi_2 = \pi_1 \\ 0.1\pi_1 + 0.2\pi_2 + 0.6\pi_3 = \pi_2 \\ 0.1\pi_1 + 0.2\pi_2 + 0.4\pi_3 = \pi_3 \end{cases} \text{ and } \pi_1 + \pi_2 + \pi_3 = 1$$

substitution $\Longrightarrow$ $\pi_1 + \pi_2 + \pi_3 = 1$ $\Longrightarrow$

$$\pi_1 = 3\pi_2, \quad \pi_2 = 1.2 \cdot \pi_3$$

$$3.6 \cdot \pi_3 + 1.2 \cdot \pi_3 + \pi_3 = 1 \Longrightarrow \pi_3 = \frac{1}{5.8}$$

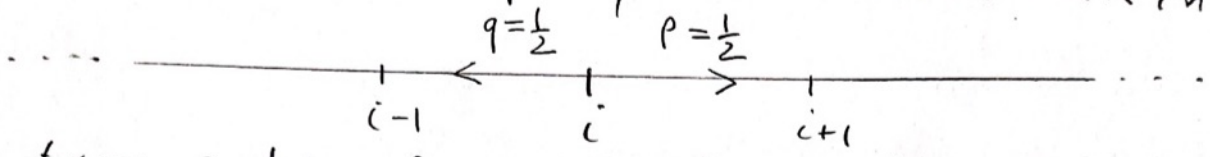
$$\Longrightarrow \pi_1 = 3.6 \frac{1}{5.8}$$

Th 3.5.7 $\Longrightarrow$

$$E[T_1] \overset{P(X_0=1)=1}{=} E[T_1 \mid X_0 = 1] \overset{\text{def}}{=} m_1 = \frac{1}{\pi_1}$$

$$= \frac{5.8}{3.6} \approx 1.61 \text{ years}$$

(54) **Ex. 3.5.9** (simple symmetric rand. walk in $\mathbb{Z}$)



→ trans. prob: $P_{i,i-1} = \frac{1}{2}$, $P_{i,i+1} = \frac{1}{2}$ (*)
(all other prob. are zero!)

→ **Problem**: Is the state zero positive recurrent, i.e. is
$E[T_0 | X_0 = 0]$
finite? (**)

Th. 3.5.7 ⇒ (**) ⟺ There ex. an invariant distr. $\pi = (\pi_i)_{i\in\mathbb{Z}}$ ($\pi_i \geq 0$, $i \in \mathbb{Z}$!), i.e.
$\pi \cdot P = \pi$ and $\sum_{i\in\mathbb{Z}} \pi_i = 1$

⟺ def. of matrix multipl. $\sum_{i\in\mathbb{Z}} \pi_i P_{ij} = \pi_j$, $j \in \mathbb{Z}$ and $\sum_{i\in\mathbb{Z}} \pi_i = 1$ (***)

Because of (*) we have for fixed j that

$$P_{ij} = \begin{cases} \frac{1}{2}, & i = j-1 \\ \frac{1}{2}, & i = j+1 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \sum_{i\in\mathbb{Z}} \pi_i P_{ij} = \pi_{j-1}\tfrac{1}{2} + \pi_{j+1}\tfrac{1}{2} \Rightarrow$$

(***) $\Leftrightarrow \pi_{j-1}\frac{1}{2} + \pi_{j+1}\frac{1}{2} = \pi_j$ for all $j \in \mathbb{Z}$ (+)
and $\sum_{j\in\mathbb{Z}} \pi_j = 1$

Assume w.l.o.g. that $\pi_0 = A > 0$

$\Rightarrow \pi_1 = A + B$ for some $B \in \mathbb{R}$

$\Rightarrow$ (+) for $j=1$: $A\cdot\frac{1}{2} + \pi_2\frac{1}{2} = A + B \Rightarrow \pi_2 = A + 2B$

Or more generally, by induction:

$\pi_j \geq 0, j\in\mathbb{Z}$ $\Rightarrow$ $B \geq 0$ $\pi_j = A + j\cdot B$, $j \geq 0$ (++)

$$\Rightarrow \sum_{j\in\mathbb{Z}} \underbrace{\pi_j}_{\geq 0} \geq \sum_{j\geq 0} \pi_j \overset{(++)}{=} \sum_{j\geq 0} (A + jB) = \infty$$

$$\Rightarrow \sum_{j\in\mathbb{Z}} \pi_j \neq 1 \Rightarrow \text{there } \underline{\text{ex. no}} \text{ invariant distribution } \pi$$

Th. 3.5.7 $\Rightarrow$ Th. 3.5.7 $E[T_0 | X_0] = \infty \Rightarrow$ zero is not positive recurrent
$\Rightarrow$ all $i \in \mathbb{Z}$ are not positive recurrent

(55) Def. for fixed $\kappa \in I$ (T_κ = first passage time of κ)

$$\gamma_i^\kappa \overset{\text{def.}}{=} E\Big[\sum_{n=0}^{T_\kappa - 1} \mathbb{1}_{\{X_n = i\}} \mid X_0 = \kappa\Big] \qquad (3.5.3)$$

$\longrightarrow$ expected time spent in i between visits to κ

Lemma 3.5.10: Let $X_n, n \geq 0$ be irreducible and recurrent with trans. matrix P. Then

(i) $\gamma_\kappa^\kappa = 1$

(ii) $\gamma^\kappa \overset{\text{def}}{=} (\gamma_i^\kappa)_{i \in I}$ fulfills $\gamma^\kappa \cdot P = \gamma^\kappa$

(iii) $0 < \gamma_i^\kappa < \infty$ for all $i \in I$

Proof: (i) $\gamma_\kappa^\kappa = 1$ follows from the def. (3.5.3)

(ii) T_κ stopping time $\overset{\text{def.}}{\Longrightarrow}$ the event $\{n \leq T_\kappa\}$ determined by $X_0, \dots, X_{n-1}$

$$P(X_n = j, X_{n-1} = i, n \leq T_\kappa \mid X_0 = \kappa) \overset{\text{def. of cond. prob.}}{=} P(X_n = j, n \leq T_\kappa, X_0 = \kappa \mid X_{n-1} = i) \cdot \frac{P(X_{n-1} = i)}{P(X_0 = \kappa)}$$

(indep. because of the Markov prop.)

$$\overset{\text{def. of indep.}}{=} \underbrace{P(X_n = j \mid X_{n-1} = i)}_{= p_{ij}} \cdot \underbrace{P(n \leq T_\kappa, X_0 = \kappa \mid X_{n-1} = i) \cdot \frac{P(X_{n-1} = i)}{P(X_0 = \kappa)}}_{= P(X_{n-1} = i, n \leq T_\kappa \mid X_0 = \kappa)}$$

$$= P(X_{n-1} = i, n \leq T_\kappa \mid X_0 = \kappa) \cdot p_{ij} \qquad (+)$$

$X_n, n \geq 0$ recurrent $\Longrightarrow P(T_\kappa < \infty \mid X_0 = \kappa) = 1 \Longrightarrow X_0 = X_{T_\kappa} = \kappa$ on $\{X_0 = \kappa\}$

$$\Longrightarrow \gamma_j^\kappa = E\Big[\sum_{n=1}^{T_\kappa} \mathbb{1}_{\{X_n = j\}} \mid X_0 = \kappa\Big] = E\Big[\sum_{n \geq 1} \mathbb{1}_{\{X_n = j, n \leq T_\kappa\}} \mid X_0 = \kappa\Big]$$

$$= \sum_{n \geq 1} E[\mathbb{1}_{\{X_n = j \text{ and } n \leq T_\kappa\}} \mid X_0 = \kappa] = \sum_{n \geq 1} P(X_n = j, n \leq T_\kappa \mid X_0 = \kappa)$$

$$= \sum_{n \geq 1} P(X_n = j, \underbrace{X_{n-1} \in I}_{= \bigcup_{i \in I} \{X_{n-1} = i\}}, n \leq T_\kappa \mid X_0 = \kappa) =$$

$$\sum_{n \geq 1} \sum_{i \in I} P(X_n = j, X_{n-1} = i, n \leq T_\kappa \mid X_0 = \kappa)$$

$$= \sum_{i \in I} \sum_{n \geq 1} \underbrace{P(X_n = j, X_{n-1} = i, n \leq T_\kappa \mid X_0 = \kappa)}_{\overset{(+)}{=} P(X_{n-1} = i, n \leq T_\kappa \mid X_0 = \kappa) \, p_{ij}} \overset{(+)}{=}$$

$$\sum_{i \in I} p_{ij} \sum_{n \geq 1} \underbrace{P(X_{n-1} = i, n \leq T_\kappa \mid X_0 = \kappa)}_{= E[\mathbb{1}_{\{X_{n-1} = i, n \leq T_\kappa\}} \mid X_0 = \kappa]}$$

$$= \sum_{i \in I} p_{ij} \, E\Big[\sum_{n \geq 1} \mathbb{1}_{\{X_{n-1} = i, n \leq T_\kappa\}} \mid X_0 = \kappa\Big]$$

$$= \sum_{i \in I} p_{ij} \, E\Big[\sum_{m=0}^{T_\kappa - 1} \mathbb{1}_{\{X_m = i\}} \mid X_0 = \kappa\Big] \overset{\text{def. of } \gamma_i^\kappa}{=} \sum_{i \in I} p_{ij} \cdot \gamma_i^\kappa \Longrightarrow \text{(i)}$$

(iii) see Th 1.7.5 in Norris $\Longrightarrow$ proof

(56) Ex. 3.5.11 : Car insurance of Ex. 3.2.7

→ X_n risk category of the customer in year n with

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0.6 & 0.4 \end{pmatrix} \quad (*)$$

→ Problem : What is the expected time spent in the highest category between visits to the lowest category ? Or more precisely γ_i^k ?

for $i=3$, $k=1$

$\xrightarrow{\text{Last Lemma}}$ $\gamma_1^1 = 1$ and $\gamma^1 = (\gamma_1^1, \gamma_2^1, \gamma_3^1)$ satisfies $\gamma^1 \cdot P = \gamma^1$

$\iff$ $\gamma_1^1 = 1$ and $\begin{cases} 0.8\,\gamma_1^1 + 0.6\,\gamma_2^1 + 0\cdot\gamma_3^1 = \gamma_1^1 \\ 0.1\,\gamma_1^1 + 0.2\,\gamma_2^1 + 0.6\,\gamma_3^1 = \gamma_2^1 \\ 0.1\,\gamma_1^1 + 0.2\,\gamma_2^1 + 0.4\,\gamma_3^1 = \gamma_3^1 \end{cases}$

$\xrightarrow{\text{substitution}}$ $\gamma_1^1 = 3\gamma_2^1$, $\gamma_2^1 = 1.2\,\gamma_3^1 \Rightarrow \underbrace{\gamma_1^1}_{=1} = 3.6 \cdot \gamma_3^1$

$\Rightarrow \gamma_3^1 = \frac{1}{3.6} \approx 0.28$ years

Proof of Th. 3.5.7 : (i) $\Rightarrow$ (ii) clear

"(ii) $\Rightarrow$ (iii)" : $\sum_{j\in J} \gamma_j^i \overset{\text{def}}{=} \sum_{j\in J} E\Big[\sum_{n=0}^{T_i-1} \mathbb{1}_{\{X_n=j\}} \,\Big|\, X_0=i\Big]$

$= E\Big[\sum_{n=0}^{T_i-1} \underbrace{\sum_{j\in J} \mathbb{1}_{\{X_n=j\}}}_{=\mathbb{1}_{\{X_n\in J\}}\equiv 1} \,\Big|\, X_0=i\Big] = E\Big[\underbrace{\sum_{n=0}^{T_i-1} 1}_{=T_i} \,\Big|\, X_0=i\Big]$

$\overset{\text{def}}{=} m_i$

$\Rightarrow \mathbb{1}_J \overset{\text{def}}{=} \frac{\gamma_j^i}{m_i}$ inv. distr.

"(iii) $\Rightarrow$ (i)" : see Norris $\Rightarrow$ Proof

3.6 Convergence to equilibrium

→ We aim at studying the n-step transition prob. $p_{ij}^{(n)}$ as $n \to \infty$

Recall: $p_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$

For this purpose we need

Def. 3.6.1 (periodic/aperiodic states)

The period of a state i, denoted by $d(i)$ is the greatest common divisor of all $n \geq 1$ s.t. $p_{ii}^{(n)} > 0$, i.e

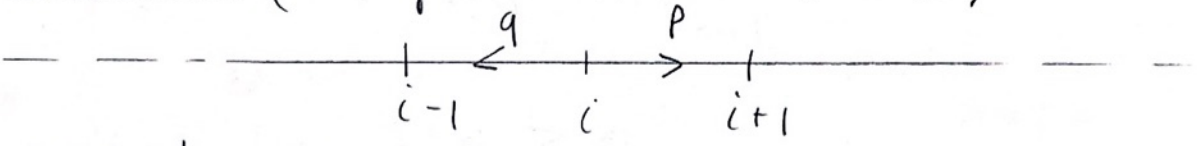
$$d(i) \overset{\text{def}}{=} g.c.d \{ n \geq 1 \,:\, p_{ii}^{(n)} > 0 \}$$

If $d(i) > 1$, we say i is periodic of period $d(i)$.

If $d(i) = 1$, then i is called aperiodic.

If $p_{ii}^{(n)} = 0$ for all $n \geq 1$, we define $d(i) = 0$

Ex. 3.6.2 (simple random walk in $\mathbb{Z}$)



$0 < p < 1$, $q := 1 - p$

We know from Ex. 3.4.2 that

$$p_{00}^{(2m+1)} = P(X_{2m+1} = 0 \mid X_0 = 0) = 0$$

and $$p_{00}^{(2m)} = P(X_{2m} = 0 \mid X_0 = 0) = \binom{2m}{m} p^m \cdot q^m > 0$$

for $m \geq 0$

$\xRightarrow[d(i) \text{ for } i=0]{\text{def. of}}$ $d(0) = 2 \implies 0$ periodic of period 2

What about the other states?

→ **Lemma 3.6.3**: If i communicates with j (i.e $i \leftrightarrow j$) then $d(i) = d(j)$

Proof: Left as an exercise (or see later)

Since all states of a random walk communicate with each other, we see from the last Lemma that

(58) all states are periodic of period 2