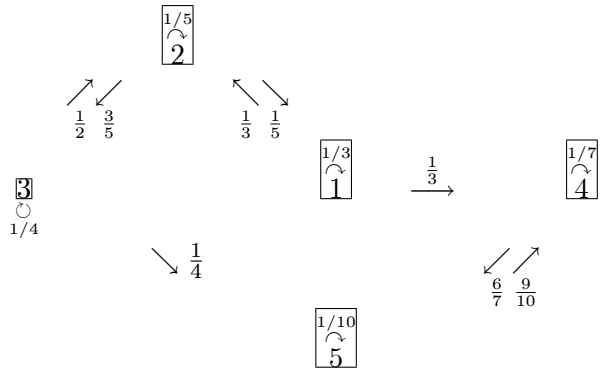
# Eksamen: Modellering av stokastiske prosesser (STK 2130)
# Løsningsforslag 11.6. 2025

**Problem 1**

(i)




(ii) As one observes from the diagram the communicating classes are given by $\{1,2,3\}$ and $\{4,5\}$. However, $\{4,5\}$ is a "trap", i.e. a closed class.

(iii)

$$P(X_2 = j \,|X_0 = i) \overset{\text{Th. 3.1.5 in the notes}}{=} p_{ij}^{(2)},$$

where

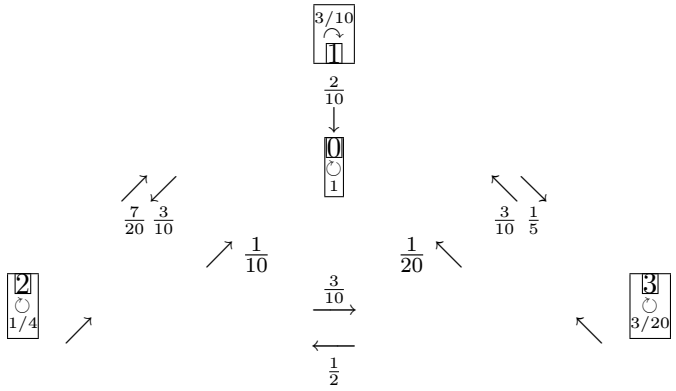
$$\begin{aligned} P^2 &= PP \overset{def}{=} (p_{ij}^{(2)})_{1\le i,j\le 5} \\ &\overset{\text{matrix mult.}}{=} \begin{pmatrix} 8/45 & 8/45 & 1/5 & 10/63 & 2/7 \\ 8/75 & 61/150 & 27/100 & 1/15 & 3/20 \\ 1/10 & 9/40 & 29/80 & 9/40 & 7/80 \\ 0 & 0 & 0 & 194/245 & 51/245 \\ 0 & 0 & 0 & 153/700 & 547/700 \end{pmatrix}. \end{aligned}$$

(iv) Finite and closed classes are recurrent. So $\{4,5\}$ is recurrent. On the other hand, recurrent classes must be closed. Hence $\{1,2,3\}$ must be transient.

We see from the diagram that $p_{55}^{(n)} > 0$ for all $n$. Hence the state $i = 5$ is aperiodic.

**Problem 2**

(i)




(ii) We want to find $h_i^0 \stackrel{def}{=} P_i(\text{hit } 0)$ for all $i = 1, 2, 3, 4$. Th. 1.3.2 in Norris shows that $h_i^0$, $i \in I = \{0, 1, 2, 3\}$ is the minimal non-negative solution of the linear equation

$$\begin{cases} h_i^0 = 1, & \text{if } i = 0 \\ h_i^0 = \sum_{j=0}^{3} p_{ij} h_j^0, & \text{if } i \neq 0 \end{cases}.$$

Using the transition probabilities we get

$$\begin{cases} h_0^0 = 1 \\ h_1^0 = 0.2 + 0.3h_1^0 + 0.3h_2^0 + 0.2h_3^0 \\ h_2^0 = 0.1 + 0.35h_1^0 + 0.25h_2^0 + 0.3h_3^0 \\ h_3^0 = 0.05 + 0.3h_1^0 + 0.5h_2^0 + 0.15h_3^0 \end{cases}.$$

Substitution of variables yields the unique (and therefore minimal) solution

$$h_1^0 = h_2^0 = h_3^0 = h_3^0 = 1.$$

(iii) By Theorem 3.2.6 in the manuscript the mean hitting times $k_i^0, i = 0, ..., 3$ are the minimal non-negative solution of

$$\begin{cases} k_i^A = 0, & \text{if } i \in A = \{0, 3\} \\ k_i^A = 1 + \sum_{j=1}^{2} p_{ij} k_j^A, & \text{if } i \in \{1, 2\} \end{cases}.$$

This implies that

$$\begin{cases} k_0^A = k_3^A = 0 \\ k_1^A = 1 + 0.3k_1^A + 0.3k_2^A \\ k_2^A = 1 + 0.35k_1^A + 0.25k_2^A \end{cases}.$$

So $k_1^A = \frac{1+0.3k_2^A}{0.7}$. Hence, using substitution gives

$$\begin{aligned} k_2^A &= \frac{4}{3}(1 + 0.35\frac{1 + 0.3k_2^A}{0.7}) = \frac{4}{3} + \frac{2}{3}(1 + 0.3k_2^A) \\ &= 2 + \frac{1}{5}k_2^A, \end{aligned}$$

which yields $k_2^A = \frac{5}{4}2 = 2.5$ weeks and $k_1^A = \frac{1+0.3\cdot 2.5}{0.7} = 2.5$ weeks.

(iv) Using the definition of expectations we get

$$\begin{aligned} & E[S_{T+2}\,|T<\infty] \\ = \;& 1\cdot P(S_{T+2}=1\,|T<\infty\,)+2\cdot P(S_{T+2}=2\,|T<\infty\,)+3\cdot P(S_{T+2}=3\,|T<\infty\,). \end{aligned}$$

Because of the strong Markov property we have that

$$P(S_{T+2}=i\,|T<\infty \text{ and } S_T=1)=P(S_2=i\,|S_0=1)=p_{1i}^{(2)}, i=0,1,2,3.$$

On the other hand the definition of $T$ implies that

$$S_T=1 \text{ with probability 1 (on } \{T<\infty\}\,).$$

So we get

$$\begin{aligned} P(S_{T+2} &= i\,|T<\infty) = P(S_{T+2}=i\,|T<\infty \text{ and } S_T=1) \\ &= P(S_2=2\,|S_0=1)=p_{1i}^{(2)}, i=0,1,2,3. \end{aligned}$$

Since

$$\begin{aligned} P^2 &= PP \stackrel{def}{=} (p_{ij}^{(2)})_{0\le i,j\le 2} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.255 & 0.265 & 0.18 \\ 0.21 & 0.2825 & 0.3175 & 0.19 \\ 0.1675 & 0.31 & 0.29 & 0.2325 \end{pmatrix} \end{aligned}$$

we obtain

$$\begin{aligned} & E[S_{T+2}\,|T<\infty] \\ = \;& p_{11}^{(2)}+2p_{12}^{(2)}+3p_{13}^{(2)}= \\ = \;& 0.255+2\cdot 0.265+3\cdot 0.18=1.325\ \$. \end{aligned}$$

**Problem 3**

1. step (see hints/formulas):

$\underline{m=0}$:
$p_{00}=P((2-D)^+=0)=P(D\ge 2)=6^{-2}=\frac{1}{36}$
$p_{01}=P((2-D)^+=1)=P(D=1)=6^{-1}-6^{-2}=\frac{5}{36}$
$p_{02}=P((2-D)^+=2)=P(D=0)=6^{-0}-6^{-1}=\frac{5}{6}$
$p_{10}=P((1-D)^+=0)=P(D\ge 1)=6^{-1}=\frac{1}{6}$
$p_{11}=P((1-D)^+=1)=P(D=0)=6^{-0}-6^{-1}=\frac{5}{6}$
$p_{12}=P((1-D)^+=2)=0$
$p_{20}=P((2-D)^+=0)=P(D\ge 2)=6^{-2}=\frac{1}{36}$
$p_{21}=P((2-D)^+=1)=P(D=1)=6^{-1}-6^{-2}=\frac{5}{36}$
$p_{22}=P((2-D)^+=2)=P(D=0)=6^{-0}-6^{-1}=\frac{5}{6}$

So we get

$$P = \begin{pmatrix} \frac{1}{36} & \frac{5}{36} & \frac{5}{6} \\ \frac{1}{6} & \frac{5}{6} & 0 \\ \frac{1}{36} & \frac{5}{36} & \frac{5}{6} \end{pmatrix}.$$

$\underline{m=1}$:
$p_{00} = P((2-D)^+ = 0) = P(D \geq 2) = 6^{-2} = \frac{1}{36}$
$p_{01} = P((2-D)^+ = 1) = P(D = 1) = 6^{-1} - 6^{-2} = \frac{5}{36}$
$p_{02} = P((2-D)^+ = 2) = P(D = 0) = 6^0 - 6^{-1} = \frac{5}{6}$
$\vdots$

So we have

$$P = \begin{pmatrix} 1/36 & 5/36 & 5/6 \\ 1/36 & 5/36 & 5/6 \\ 1/36 & 5/36 & 5/6 \end{pmatrix}.$$

2. step: Invariant distribution
$\underline{m=0}$: $\pi P = \pi$ becomes

$$\begin{cases} \pi_0 \frac{1}{36} + \pi_1 \frac{1}{6} + \pi_2 \frac{1}{36} = \pi_0 \\ \pi_0 \frac{5}{36} + \pi_1 \frac{5}{6} + \pi_2 \frac{5}{36} = \pi_1 \\ \pi_0 \frac{5}{6} + \pi_1 0 + \pi_2 \frac{5}{6} = \pi_2 \end{cases}$$

$\Longrightarrow \pi_0 = \frac{1}{5}\pi_2 \Longrightarrow \frac{1}{36}\pi_2 + \pi_1 \frac{5}{6} + \pi_2 \frac{5}{36} = \pi_1 \Longrightarrow \pi_2 = \pi_1 \overset{\pi_0+\pi_1+\pi_2=1}{\Longrightarrow} \frac{1}{5}\pi_2 + \pi_2 + \pi_2 = 1 \Longrightarrow \pi_2 = \frac{5}{11}$. So

$$\pi = (\frac{1}{11}, \frac{5}{11}, \frac{5}{11}).$$

$\underline{m=1}$: It's easy to guess that

$$\pi = (\frac{1}{36}, \frac{5}{36}, \frac{5}{6}).$$

3. step (see hints/formulas):

$$\begin{aligned} E\left[(D-j)^+\right] &= \sum_{k\geq 1} P((D-j)^+ \geq k) = \sum_{k\geq 1} P(D \geq k+j) \\ &= \sum_{k\geq 1} 6^{-(k+j)} = 6^{-j}(\sum_{k\geq 1} 6^{-k}) = 6^{-j}(\frac{1}{1-\frac{1}{6}} - 1) \\ &= 6^{-j}\frac{1}{5}. \end{aligned}$$

Hence we get
$\underline{m=0}$: $f(0) = E[(D-2)^+] \overset{j=2}{=} \frac{1}{180}$, $f(1) = E[(D-1)^+] = \frac{1}{30}$, $f(2) = E[(D-2)^+] = \frac{1}{180}$.

$\underline{m=1}$: $f(0) = f(1) = f(2) = E[(D-2)^+] = \frac{1}{180}$.

4. step (see hints/formulas): Calculation of the cost functional

$\underline{m=0}$: $\overline{f}(0) = \sum_{i=0}^{2} \pi_i f(i) = \frac{1}{11}\frac{1}{180} + \frac{5}{11}\frac{1}{30} + \frac{5}{11}\frac{1}{180} = \frac{1}{55}$ and $\overline{g}(0) = \sum_{i=0}^{2} \pi_i g(i) = \pi_0 = \frac{1}{11}$.

$\underline{m=1}$: $\overline{f}(1) = \sum_{i=0}^{2} \pi_i f(i) = \frac{1}{36}\frac{1}{180} + \frac{5}{36}\frac{1}{180} + \frac{5}{6}\frac{1}{180} = \frac{1}{180}$ and $\overline{g}(1) = \sum_{i=0}^{2} \pi_i g(i) = \pi_0 + \pi_1 = \frac{1}{36} + \frac{5}{36} = \frac{1}{6}$.

So $m = 0$ is minimal if

$$\begin{aligned} a\overline{g}(0) + b\overline{f}(0) &= \frac{3}{11} + \frac{b}{55} \\ &\leq a\overline{g}(1) + b\overline{f}(1) = \frac{3}{6} + \frac{b}{180}. \end{aligned}$$

The latter is equivalent to saying that

$$b \leq 18.$$

On the other hand $m = 1$ is minimal if

$$b \geq 18.$$

**Problem 4** In order to apply the dichotomy theorem for Markov chains, we first want to calculate

$$P(T_0 < \infty \,|X_0 = 0)$$

where

$$T_0 := \inf\{n \geq 1 : X_n = 0\}.$$

Because of the Markov property we have that

$$\begin{aligned} P(T_0 &< \infty \,|X_1 = i) = P(\inf\{n \geq 0 : X_{n+1} = 0\} + 1 < \infty \,|X_1 = i) \\ &= P(H^0 < \infty \,|X_0 = i) = h_i^0, i \geq 0. \end{aligned} \tag{1}$$

On the other hand we have that

$$\begin{aligned} P(T_0 &< \infty \,|X_0 = 0) = \frac{P(T_0 < \infty, X_0 = 0)}{P(X_0 = 0)} = \sum_{i \geq 0} \frac{P(T_0 < \infty, X_0 = 0, X_1 = i)}{P(X_0 = 0)} \\ &= \sum_{i \geq 0} P(T_0 < \infty, X_0 = 0 \,|X_1 = i)\, \frac{P(X_1 = i)}{P(X_0 = 0)} \\ &\overset{\text{Markov property}}{=} \sum_{i \geq 0} P(T_0 < \infty \,|X_1 = i)\, P(X_0 = 0 \,|X_1 = i)\, \frac{P(X_1 = i)}{P(X_0 = 0)} = \sum_{i \geq 0} P(T_0 < \infty \,|X_1 = i)\, p_{0i} \\ &= P(T_0 < \infty \,|X_1 = 1) \overset{(1)}{=} P(H^0 < \infty \,|X_0 = 1) = h_1^0. \end{aligned} \tag{2}$$

We know that $h_i^0, i \geq 0$ is the minimal non-negative solution to the equation

$$\begin{cases} h_0^0 = 1 & \text{if } i = 0 \\ h_i^0 = \sum_{j \geq 0} p_{ij} h_j^0 & \\ \quad = p_{i,i-1} h_{i-1}^0 + p_{i,i+1} h_{i+1}^0 & \text{if } i \neq 0 \end{cases}. \tag{3}$$

Set

$$u_i = h^0_{i-1} - h^0_i,\ p_i = p_{i,i+1} \text{ and } q_i = p_{i,i-1}, i \geq 1.$$

Then $p_i u_{i+1} = q_i u_i$, so

$$u_{i+1} = \left(\frac{q_i}{p_i}\right) u_i = \left(\frac{q_i q_{i-1} ... q_1}{p_i p_{i-1} ... p_1}\right) u_1 = \gamma_i u_1,$$

where

$$\gamma_i := \frac{q_i q_{i-1} ... q_1}{p_i p_{i-1} ... p_1}.$$

Then

$$u_1 + ... + u_i = h^0_0 - h^0_i$$

and so

$$h^0_i = 1 - A(\gamma_0 + ... + \gamma_{i-1}), \tag{4}$$

where $A = u_1$ (unknown) and $\gamma_0 := 1$.

We see that

$$\gamma_i = (\prod_{j=1}^{i} \frac{j}{j+1})^4 = \frac{1}{(i+1)^4}, i \geq 1.$$

So $\sum_{i \geq 0} \gamma_i < \infty$. Therefore the minimal non-negative solution in (4) is given for

$$A = 1/(\sum_{i \geq 0} \gamma_i) \overset{\text{hints/formulas}}{=} 90/\pi^4 \approx 0.924.$$

So from (1) and (3) we see that

$$P(T_0 < \infty \,|X_0 = 0) = 1 - 90/\pi^4 \approx 0.076 < 1.$$

It follows from the dichotomy theorem (see Norris) that 0 is a transient state.

Further, let $j > i$. Then

$$\begin{aligned} P(X_m &= j \,|X_0 = i) = p^{(m)}_{ij} = \sum_{i_1,...,i_{m-1} \in I} p_{ii_1} p_{i_1 i_2} ... p_{i_{m-1} j} \\ &\geq p_{ii+1} p_{i+1 i+2} ... p_{j-1 j} > 0. \end{aligned}$$

Similarly, one shows that $P(X_m = j \,|X_0 = i) > 0$, when $i > j$. So $I$ is a communicating class. The latter implies that states of $I$ are transient.

Define $R_k = \{0, ..., k\}$ and $L_k = \{X_n \in R_k \text{ for infinitely many } n\}$. Then the complement of the event $L_k$ is given by

$$L^c_k = \{\text{there exists a } N \text{ such that for all } n \geq N :\ X_n \notin R_k\}.$$

We also observe that $L^c_{k+1} \subseteq L^c_k$. Using the latter and the observation that $\{\lim_{n \longrightarrow \infty} X_n = \infty\} = \cap_{k \geq 0} L^c_k$,we find that

$$P(\lim_{n \longrightarrow \infty} X_n = \infty) = P(\cap_{k \geq 0} L^c_k) = \lim_{k \longrightarrow \infty} P(L^c_k).$$

We see that $P(L^c_k) = 1 - P(L_k)$. On the other hand, $L_k \subseteq \cup^k_{j=0} \{X_n = j \text{ for infinitely many } n\}$. So the transience of the states implies that $P(L_k) = 0$ and therefore we obtain that $P(\lim_{n \longrightarrow \infty} X_n = \infty) = 1$.