UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	STK2130 — Modeling of Stochastic Processes.
Day of examination:	June 11, 2025, Final written exam.
Examination hours:	09:00 – 13:00.
This problem set consists of 4 pages.	
Appendices:	None
Permitted aids:	Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1. (10 points)

Consider a Markov chain $X_n, n \geq 0$ with state space $I = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{7} & \frac{6}{7} \\ 0 & 0 & 0 & \frac{9}{10} & \frac{1}{10} \end{pmatrix}.$$

(i) Describe the Markov chain by a diagram.

(ii) Which are the communicating classes of the Markov chain ? Which are closed ?

(iii) Compute for all $i, j = 1, ..., 5$ the probability that $X_2 = j$ given $X_0 = i$.

(iv) Determine the recurrent and transient states. What is the period of the state $i = 5$?

Problem 2. (10 points)

Consider stock price process S_n with values

$$S_n = 0 \text{ \$ or } 1 \text{ \$ or } 2 \text{ \$ or } 3 \text{ \$}$$

for all $n = 0, 1, 2, ...$(in weeks).
Suppose that $(S_n)_{n \geq 0}$ is a Markov chain with stochastic matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.3 & 0.2 \\ 0.1 & 0.35 & 0.25 & 0.3 \\ 0.05 & 0.3 & 0.5 & 0.15 \end{pmatrix}.$$

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(i) Describe the Markov chain by a diagram.

(ii) Calculate for all $i = 0, 1, 2, 3$ the probability that the stock price starting in i \$ ever reaches 0 \$.

(iii) Find for all $i = 0, 1, 2, 3$ the mean time for the stock price hitting $A = \{0, 3\}$ given that its initial value is i (\$).

(iv) A trader in a market, who observes the stock prices $S_n, n = 0, 1, 2, ...$, decides to buy the stock, if it costs 1 \$.

Compute the expected value of the stock price two weeks after the purchase, that is

$$E[S_{T+2} \,|T < \infty] \,.$$

Here $T = H^A$, that is the hitting time

$$H^A = \inf\{n \geq 0 : S_n \in A\},$$

where $A = \{1\}$.

Problem 3.(Restocking of a warehouse) (10 points)

Consider a warehouse with a capacity of c units of a stock. Let D_n be the demand for the stock in time period n. In addition, denote by X_n the residual stock at the end of time period n. The warehouse manager decides to restock to capacity c in the beginning of time period $n + 1$, if

$$X_n \leq m,$$

where $m \in \{0, 1, ..., c - 1\}$ is a certain threshold. Hence, $X_n, n \geq 0$ is given by

$$X_{n+1} = \begin{cases} (c - D_{n+1})^+ & \text{if } X_n \leq m \\ (X_n - D_{n+1})^+ & \text{if } m < X_n \leq c \end{cases},$$

where $(a)^+ \stackrel{def}{=} \max(a, 0)$ for $a \in \mathbb{R}$.
Assume that $D_n, n \geq 1$ are independent and identically $(i.i.d)$ distributed random variables with common distribution D given by

$$P(D \geq i) = 6^{-i},$$

that is $P(D = i) = 6^{-i} - 6^{-(i+1)}, i \geq 1$ $(D \in \{0, 1, 2, ...\})$. In addition, assume that $c = 2$ (i.e. $m \in \{0, 1\}$).

Find the optimal restocking strategy of the warehouse manager, that is minimize the cost function

$$a\overline{g}(m) + b\overline{f}(m)$$

with respect to m, where $\overline{g}(m)$ stands for the long-run frequency of restocking and where $\overline{f}(m)$ is the long-run proportion of unmet demand. Further, $a \geq 0$ is the cost for restocking

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and $b \geq 0$ is the (lost) profit with respect to unmet demand per unit stock. Assume that $a = 3$ (in units of a given currency).

Problem 4. (10 points)

Consider a Markov chain $X_n, n \geq 0$ on the state space $I = \{0, 1, 2, ...\}$ with transition probabilities given by

$$p_{01} = 1,\ p_{i,i+1} + p_{i,i-1} = 1, p_{i,i+1} = \left(\frac{i+1}{i}\right)^4 p_{i,i-1}, i \geq 1.$$

Compute the probability

$$P(\lim_{n \longrightarrow \infty} X_n = \infty).$$

Hints and some formulas

a) $h_i^A, i \in I$ (hitting probability) is the minimal non-negative solution to

$$\begin{cases} h_i^A = 1 & \text{if } i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & \text{if } i \notin A \end{cases}.$$

b) $k_i^A, i \in I$ (mean hitting time) is the minimal non-negative solution to

$$\begin{cases} k_i^A = 0 & \text{if } i \in A \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A & \text{if } i \notin A \end{cases}.$$

c) (i)

$$\overline{f}(m) \stackrel{def}{=} \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(X_j) = \sum_{i=0}^{c} \pi_i f(i),$$

where $\pi = (\pi_0, ..., \pi_c)$ is the invariant distribution and

$$f(i) \stackrel{def}{=} E\left[\widetilde{X}_{n+1} \,|X_n = i\right] \ = \ \begin{cases} E\left[(D-c)^+\right] & \text{if } i \leq m \\ E\left[(D-i)^+\right] & \text{if } m < i \leq c \end{cases}$$

(expected unmet demand)

(ii)

$$\sum_{k \geq 0} q^k = \frac{1}{1-q},\ 0 \leq q < 1$$

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(iii)

$$\overline{g}(m) \stackrel{def}{=} \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(X_j) = \sum_{i=0}^{c} \pi_i g(i),$$

where $\pi = (\pi_0, ..., \pi_c)$ is the invariant distribution and

$$g(i) \stackrel{def}{=} \begin{cases} 1 & \text{if } 0 \leq i \leq m \\ 0 & \text{else} \end{cases}.$$

(iv)

$$p_{ij} = \begin{cases} P((c-D)^+ = j) & \text{if } 0 \leq i \leq m \\ P((i-D)^+ = j) & \text{if } m < i \leq c \end{cases}.$$

d)

$$\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

e) (i) $Y \in \{0, 1, 2, ...\}$ random variable $\Longrightarrow$

$$E[Y] = \sum_{k \geq 1} P(Y \geq k).$$

(ii) $A := \cup_{i \geq 1} A_i$ with events $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ Then

$$P(A) = \lim_{i \longrightarrow \infty} P(A_i).$$

End