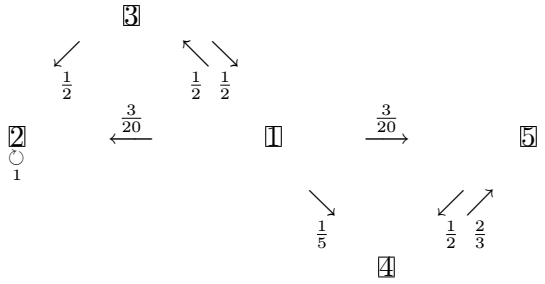
Prøveeksamen (STK 2130)
Løsningsforslag

Problem 1

(i)



(ii) As one observes from the diagram the communicating classes are given by $\{1,3\},\{2\}$ and $\{4,5\}$. However, $\{2\}$ and $\{4,5\}$ are "traps", i.e. a closed classes.

(iii)

$$P(X_2 = j\,|X_0 = i) \overset{\text{Th. 3.1.5 in the notes}}{=} p_{ij}^{(2)},$$

where

$$\begin{aligned} P^2 &= PP \overset{def}{=} (p_{ij}^{(2)})_{1\le i,j\le 5} \\ &\overset{\text{matrix mult.}}{=} \begin{pmatrix} 0.25 & 0.4 & 0 & 0.142 & 0.208 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0.575 & 0.25 & 0.1 & 0.075 \\ 0 & 0 & 0 & 0.444 & 0.556 \\ 0 & 0 & 0 & 0.417 & 0.583 \end{pmatrix}. \end{aligned}$$

(iv) Finite and closed classes are recurrent. So $\{2\}$ and $\{4,5\}$ are recurrent. On the other hand, recurrent classes must be closed. Hence $\{1,3\}$ must be transient.

We see from the diagram that $p_{11}^{(2n+1)} = 0$ for all n. On the other hand, $p_{11}^{(2n)} > 0$ for all n. So the period of $i = 1$ is 2.

Problem 3

(i) We want to find $h_i^0 \overset{def}{=} P_i(\text{hit } 0)$ for all $i = 1,2,3$. Th. 1.3.2 in Norris shows that h_i^0, $i \in I = \{0,1,2\}$ is the minimal non-negative solution of the linear equation

$$\begin{cases} h_i^0 = 1, & \text{if } i = 0 \\ h_i^0 = \sum_{j=0}^{2} p_{ij} h_j^0, & \text{if } i \neq 0 \end{cases}.$$

Using the transition probabilities we get

$$\begin{cases} h_0^0 = 1 \\ h_1^0 = 0.2 + 0.48 h_1^0 + 0.32 h_2^0 \\ h_2^0 = 0.1 + 0.6 h_1^0 + 0.3 h_2^0 \end{cases}.$$

Substitution of variables yields the unique (and therefore minimal) solution

$$h_1^0 = h_2^0 = h_3^0 = 1.$$

(ii) By Theorem 3.2.6 in the manuscript the mean hitting times $k_i^0, i = 0, ..., 3$ are the minimal non-negative solution of

$$\begin{cases} k_i^0 = 0, & \text{if } i = 0 \\ k_i^0 = 1 + \sum_{j=1}^{2} p_{ij} k_j^0, & \text{if } i \neq 0 \end{cases}.$$

This implies that

$$\begin{cases} k_0^0 = 0 \\ k_1^0 = 1 + 0.48 k_1^0 + 0.32 k_2^0 \\ k_2^0 = 1 + 0.6 k_1^0 + 0.3 k_2^0 \end{cases}.$$

So using substitution gives the unique solution

$$k_0^0 = 0, k_1^0 = 5.930, k_2^0 = 6.512 \text{ in weeks.}$$

(iii) Using the definition of expectations we get

$$E[S_{T+3} \,|T < \infty] = 1 \cdot P(S_{T+3} = 1 \,|T < \infty\,) + 2 \cdot P(S_{T+3} = 2 \,|T < \infty\,).$$

Because of the strong Markov property we have that

$$P(S_{T+3} = i \,|T < \infty \text{ and } S_T = 1) = P(S_3 = i \,|S_0 = 1) = p_{1i}^{(3)}, i = 0, 1, 2.$$

On the other hand the definition of T implies that

$$S_T = 1 \text{ with probability 1 (on } T < \infty).$$

So we get

$$\begin{aligned} P(S_{T+3} &= 2 \,|T < \infty) = P(S_{T+3} = 2 \,|T < \infty \text{ and } S_T = 1) \\ &= P(S_3 = 2 \,|S_0 = 1) = p_{12}^{(3)}. \end{aligned}$$

Since

$$\begin{aligned} P^3 &= PPP \stackrel{def}{=} (p_{ij}^{(3)})_{0 \leq i,j \leq 2} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0.437 & 0.353 & 0.210 \\ 0.372 & 0.394 & 0.234 \end{pmatrix} \end{aligned}$$

we obtain

$$E[S_{T+3} \,|T < \infty] = p_{11}^{(3)} + 2 p_{12}^{(3)} = 0.773 \text{ \$}.$$