# Prøveeksamen (STK2130)

**Prob. 4:** Warehouse with a capacity of $c$ units of stock

$D_n$ demand for the stock in units at time period $n$

→ $X_n$ residual stock at the end of period $n$

The manager restocks to capacity $c$ at the beginning of $n+1$, if

$$X_n \le m \quad (m \in \{0, \dots, c-1\})$$

$$\rightarrow \quad X_{n+1} = \begin{cases} (c - D_{n+1})^+, & X_n \le m \\ (X_n - D_{n+1})^+, & m < X_n \le c \end{cases} \qquad (1)$$

$(a)^+ \overset{\text{def}}{=} \max(0, a)$

$D_n, n \ge 1$ i.i.d with common law $D$ given by

$$\Rightarrow \quad P(D = i) = P(D \ge i) - P(D \ge i+1) = 4^{-i} - 4^{-(i+1)} \qquad (2)$$

$c = 2$

**Objective:** $a\,\bar{g}(m) + b\,\bar{f}(m) \rightarrow \min$ w.r.t. $m$

for $a = 2$.

$$\bar{f}(m) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(X_j) \overset{\text{Ergodic Theorem}}{=} \sum_{i=0}^{c} \pi_i f(i)$$

($\pi$: inv. distr.)

$$f(i) \overset{\text{def.}}{=} E\left[\tilde{X}_{n+1} \mid X_n = i\right] \qquad (3)$$

$$\tilde{X}_{n+1} = \begin{cases} (D_{n+1} - c)^+, & X_n \le m \\ (D_{n+1} - X_n)^+, & m < X_n \le c \end{cases}$$

→ $\tilde{X}_{n+1}$ unmet demand at $n+1$

→ $\bar{f}(m)$ long-run proportion of unmet demand

$$\bar{g}(m) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(X_j) \overset{\text{Ergodic Theorem}}{=} \sum_{i=0}^{c} \pi_i g(i)$$

$$g(i) \overset{\text{def}}{=} \mathbb{1}_{\{i \le m\}} = \begin{cases} 1, & i \le m \\ 0 & \text{else} \end{cases}$$

→ $\bar{g}(m)$ long-run frequency of restocking

$a \triangleq$ cost of restocking per unit stock

$b \triangleq$ (lost) profit per unit stock

Prøveeksamen

② $\overline{g}(m)$, $\overline{P}(m)$ ? for $m = 0, 1$

→ 1. step: (computation of $P_{ij}$ for $m = 0, 1$

$P_{ij} \overset{\text{def}}{=} P(X_{n+1} = j \mid X_n = i) \overset{(1)}{=} \overline{D}_n$ i.i.d

$$\begin{cases} P((2-D)^+ = j), & i \leq m \\ P((i-D)^+ = j), & m < i \leq 2 \end{cases} \quad (4)$$

See Ex. 4.2.1

⇒ $m = 0$:

$P_{00} \overset{(4)}{=} P((2-D)^+ = 0) = P(D \geq 2) \overset{(2)}{=} 4^{-2} = \frac{1}{16}$

$P_{01} = P((2-D)^+ = 1) = P(D = 1) \overset{(2)}{=} 4^{-1} - 4^{-2} = \frac{3}{16}$

$P_{02} = P((2-D)^+ = 2) = P(D = 0) = 4^{-0} - 4^{-1} = \frac{3}{4}$

$P_{10} \overset{(4)}{=} P((1-D)^+ = 0) = P(D \geq 1) = 4^{-1} \overset{(2)}{=} \frac{1}{4}$

$P_{11} \overset{(4)}{=} P((1-D)^+ = 1) = P(D = 0) = 4^{-0} - 4^{-1} = \frac{3}{4}$

$P_{12} \overset{(4)}{=} P((1-D)^+ = 2) = 0$

$P_{20} \overset{(4)}{=} P((2-D)^+ = 0) = P(D \geq 2) = 4^{-2} = \frac{1}{16}$

$P_{21} = P((2-D)^+ = 1) = P(D = 1) = 4^{-1} - 4^{-2} = \frac{3}{16}$

$P_{22} = P((2-D)^+ = 2) = P(D = 0) = 4^{-0} - 4^{-1} = \frac{3}{4}$

$$\longrightarrow \quad P = \begin{pmatrix} 1/16 & 3/16 & 3/4 \\ 1/4 & 3/4 & 0 \\ 1/16 & 3/16 & 3/4 \end{pmatrix}$$

$m = 1$: $P_{00} \overset{(4)}{=} P((2-D)^+ = 0) = P(D \geq 2) \overset{(2)}{=} 4^{-2} = \frac{1}{16}$

$$\longrightarrow \quad P = \begin{pmatrix} 1/16 & 3/16 & 3/4 \\ 1/16 & 3/16 & 3/4 \\ 1/16 & 3/16 & 3/4 \end{pmatrix}$$

2. step: (computation of $\Pi = (\Pi_0, \Pi_1, \Pi_2)$

$m = 0$: $\Pi$ inv. distr. $\Longleftrightarrow$ $\Pi \cdot P = \Pi$ and $\Pi_0 + \Pi_1 + \Pi_2 = 1$

$\Longleftrightarrow$

$$\begin{cases} \Pi_0 \frac{1}{16} + \Pi_1 \frac{1}{4} + \Pi_2 \frac{1}{16} = \Pi_0 \\ \Pi_0 \frac{3}{16} + \Pi_1 \frac{3}{4} + \Pi_2 \frac{3}{16} = \Pi_1 \\ \Pi_0 \frac{3}{4} + \Pi_1 \cdot 0 + \Pi_2 \frac{3}{4} = \Pi_2 \end{cases} \quad \text{and } \Pi_0 + \Pi_1 + \Pi_2 = 1$$

$\Longrightarrow \Pi_0 \frac{3}{4} = \frac{1}{4} \Pi_2 \Longrightarrow \Pi_0 = \frac{1}{3} \Pi_2$

$\Longrightarrow \frac{1}{4} \Pi_2 = \frac{1}{4} \Pi_1 \Longrightarrow \Pi_2 = \Pi_1$

$\Longrightarrow \frac{1}{3} \Pi_2 + \Pi_2 + \Pi_2 = 1 \Longrightarrow \Pi_2 = \frac{3}{7}$

$\Longrightarrow \Pi = (\Pi_0, \Pi_1, \Pi_2) = \left(\frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right)$

$m = 1$: $\Pi = \left(\frac{1}{16}, \frac{3}{16}, \frac{3}{4}\right)$

(3)

**3. step**: Comp. of $f(i)$, $i=0,1,2$

$$f(i) \overset{(3)}{=} D_n \text{ i.i.d} \begin{cases} E[(D-2)^+], & i \le m \\ E[(D-i)^+], & m < i \le 2 \end{cases} \quad (5)$$

See Ex. 4.2.1

$$E[(D-j)^+] = \sum_{K\ge 1} P((D-j)^+ \ge K) = \sum_{K\ge 1} \underbrace{P(D \ge K+j)}_{(2)\; 4^{-(K+j)}} \overset{(2)}{=} \sum_{K\ge 1} 4^{-(K+j)} = 4^{-j}\left(\sum_{K\ge 1} 4^{-K}\right) \quad (6)$$

$$\sum_{K\ge 1} 4^{-K} = \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3}$$

Recall: (i) $E[Y] = \sum_{K\ge 1} P(Y \ge K)$, $Y \in \{0,1,2,\dots\}$

(ii) $\sum_{K\ge 0} q^K = \frac{1}{1-q}$, $0 \le q < 1$

$m=0$:

$$f(0) \overset{(5)}{=} E[(D-2)^+] \overset{(6)}{=} 4^{-2}\cdot\frac{1}{3} = \frac{1}{48}$$
$$f(1) \overset{(5)}{=} E[(D-1)^+] \overset{(6)}{=} 4^{-1}\cdot\frac{1}{3} = \frac{1}{12}$$
$$f(2) \overset{(5)}{=} E[(D-2)^+] = \frac{1}{48}$$

$m=1$:

$$f(0) = f(1) = f(2) = E[(D-2)^+] \overset{(6)}{=} 4^{-2}\frac{1}{3} = \frac{1}{48}$$

**4. step**: Comp. of $\bar{f}(m)$, $\bar{g}(m)$, $m=0,1$

$m=0$:

$$\bar{f}(0) \overset{\text{def}}{=} \sum_{i=0}^{2} \pi_i f(i) \overset{2.,3.\text{step}}{=} \frac{1}{7}\frac{1}{48} + \frac{3}{7}\frac{1}{12} + \frac{3}{7}\frac{1}{48} = \frac{1}{21}$$

$$\bar{g}(0) \overset{\text{def}}{=} \sum_{i=0}^{2} \pi_i \mathbb{1}_{\{i\le 0\}} = \pi_0 \overset{2.\text{step}}{=} \frac{1}{7}$$

$$\Rightarrow 2 \overset{a}{=} \bar{g}(0) + b\,\bar{f}(0) = \frac{2}{7} + \frac{b}{21}$$

$m=1$:

$$\bar{f}(1) \overset{\text{def}}{=} \sum_{i=0}^{2} \pi_i f(i) \overset{2.,3.\text{step}}{=} \frac{1}{16}\cdot\frac{1}{48} + \frac{3}{16}\frac{1}{48} + \frac{3}{4}\cdot\frac{1}{48} = \frac{1}{48}$$

$$\bar{g}(1) \overset{\text{def}}{=} \sum_{i=0}^{2} \pi_i \mathbb{1}_{\{i\le 1\}} = \pi_0 + \pi_1 \overset{2.\text{step}}{=} \frac{1}{16} + \frac{3}{16} = \frac{1}{4}$$

$$\Rightarrow 2 \overset{a}{=} \bar{g}(1) + b\,\bar{f}(1) = \frac{2}{4} + \frac{b}{48}$$

$\Rightarrow$ costs become for $m=0$ minimal $\iff$

$$\frac{2}{7} + \frac{b}{21} \le \frac{2}{4} + \frac{b}{48} \iff \frac{b}{21} - \frac{b}{48} \le \frac{1}{2} - \frac{2}{7} \iff \underline{b \le 8}$$

$\|$ $\|$ for $m=1$ minimal $\iff$

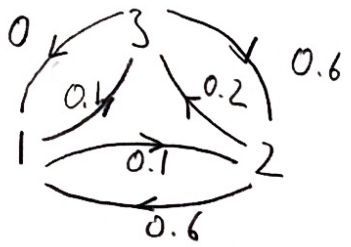
$$\frac{2}{4} + \frac{b}{48} \le \frac{2}{7} + \frac{b}{21} \iff \underline{b \ge 8}$$

(4) 2. Prob: Consider the car insurance of Ex. 3.2.7

→ $X_n \in \mathcal{I} = \{1,2,3\}$ risk category of the insured in year $n$

$X_n, n \geq 0$ Markov$(\lambda, P)$ with

$$P = \begin{pmatrix} 0.7 & 0.15 & 0.15 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0.6 & 0.4 \end{pmatrix}$$




$K_i^A$ 2

for $i=1,2,3$ and $A=\{1\}$

$K_i^A \overset{\text{def}}{=} E[H^A \mid X_0 = i]$

→ mean time that the chain starting in $i$ ever hits $A=\{1\}$

$H^A := \inf\{n \geq 0 : X_n \in A\}$ hitting time

→ first time that $X_n$ hits $A$

→ $K_i^A$ mean time that the customer starting in risk category $i$ ever enters the lowest risk category

By Th. 3.2.6 $K_i^A$, $i \in \mathcal{I} = \{1,2,3\}$ is the min. non-neg. solution to the equations

$$\begin{cases} K_i^A = 0, & i \in A \\ K_i^A = 1 + \sum_{j \notin A} p_{ij} K_j^A, & i \notin A \end{cases}$$

$A=\{1\}$ ⟺

$$\begin{cases} K_1^A = 0 \\ K_2^A = 1 + \underbrace{p_{22}}_{=0.2} K_2^A + \underbrace{p_{23}}_{=0.2} K_3^A \\ K_3^A = 1 + \underbrace{p_{32}}_{=0.6} K_2^A + \underbrace{p_{33}}_{=0.4} K_3^A \end{cases}$$

$$\Rightarrow \quad 0.8 K_2^A = 1 + 0.2 \cdot K_3^A \quad | \cdot 3$$

$$0.6 K_3^A = 1 + 0.6 K_2^A$$

$$\Rightarrow \quad 2.4 K_2^A = 3 + 0.6 K_3^A$$

$$0.6 K_3^A = 1 + 0.6 K_2^A$$

$$\Rightarrow \quad 2.4 K_2^A = 3 + 1 + 0.6 K_2^A \iff K_2^A = \frac{4}{1.8} \approx \underline{2.222 \text{ years}}$$

$$\Rightarrow \quad K_3^A = \frac{1}{0.6}\left(1 + 0.6 \cdot \frac{4}{1.8}\right) \approx \underline{3.889 \text{ years}}$$

(5) period of $i = 1, 2, 3$ 2

$$P_{ii}^{(n)} = P(X_n = i \mid X_0 = i)$$

$$\geqslant_{\text{Th. 3.1.3}} P(X_0 = i, X_1 = i, X_2 = i, \ldots, X_n = i \mid X_0 = i)$$

$$= P_{ii} P_{ii} \cdots P_{ii} > 0$$

$$>0 \quad >0 \quad >0$$

for all $n$

$\Rightarrow$ $i = 1, 2, 3$ aperiodic